

Stirling+Fedoruk

Famous Stirling formula is a good problem which explains stationary phase method including its non-trivial part. I deduce standard Stirling formula using Ted Voronov's comments on Fedoruk paper

I returned to this text again in June 2020...(see another version in math Blog on October 2019)

Method of stationary phase... Almost everybody knows that it is about how to calculate the leading term in in the integral

$$\int \exp \frac{iS(x)}{\hbar} u(x) dx$$

if $\hbar \rightarrow 0$. If x_0 is unique stationary point of function S ,

$$S(x) = S(x_0) + A_{ab}(x^a - x_0^a)(x^b - x_0^b) + S_+(x)$$

and Hessian at this point is not degenerate, then

$$\int \exp \frac{iS(x)}{\hbar} u(x) dx = C \exp \frac{iS(x_0)}{\hbar} \frac{(\pi\hbar)^{\frac{n}{2}}}{\sqrt{\det A}} (1 + O(\hbar)) . \tag{01}$$

People say that if $\hbar \rightarrow 0$ then the main contribution to the integral is given by vicinity of the stationary point, and this is just gaussian integral.

What about other terms of expansion?

There is deep and powerfull statement: All other terms are polynomial in \hbar ! Namely

$$\int \exp \frac{iS(x)}{\hbar} u(x) dx = C \exp \frac{iS(x_0)}{\hbar} \frac{(\pi\hbar)^{\frac{n}{2}}}{\sqrt{\det A}} \underbrace{\left(\exp \left(-\frac{\hbar}{2i} A^{ab} \frac{\partial}{\partial x^a} \frac{\partial}{\partial x^b} \right) \right)}_I \left(\exp \frac{iS_+(x)}{\hbar} \right) \tag{02}$$

whereas the expression I is an expansion over positive powers of \hbar ., i.e.

$$\text{Only positive powers of } \hbar \text{ have input in } O(\hbar) \tag{03}$$

On one hand everybody knows the statement (01), on the other way the statement (02) is not simple, and it is highly non-trivial.

The best way to see this statement it is to study famous Stirling formula.

Recall that Stirling formula says that

$$N! = \left(\frac{N}{e} \right)^N \sqrt{2\pi N} \left(1 + \frac{1}{12N} + \frac{1}{288N^2} + \dots \right) \tag{Stirling}$$

The "easy part" of Stirling formula, i.e. the statement

$$N! \approx \left(\frac{N}{e}\right)^N \sqrt{2\pi N}, \quad N! = \left(\frac{N}{e}\right)^N \sqrt{2\pi N} \left(1 + O\left(\frac{1}{N}\right)\right),$$

is related with (01). Equation (Stirling) is the 'difficult part'. of Stirling formula. In particular studying this equation it is impossible to avoid the question: why in formal series appear only negative powers of N (compare with (02,03)).

Calculations of Stirling formula

This is standard that

$$N! = \int_0^\infty t^N e^{-t} dt = \int_0^\infty e^{-t+N \log t} dt.$$

Calculate it.

To use stationary phase method consider substitution $t = N(1+x)$ (The point $t = -N$ is stationary point, we consider $t = N(1+x)$ not just $t = 1+x$ to have 'big coefficient' in a vicinity of stationary point)

Thus we have:

$$\begin{aligned} N! &= \int_0^\infty t^N e^{-t} dt = \int_0^\infty e^{-t+N \log t} dt = \int_{-1}^\infty e^{-N(1+x)+N \log[N(1+x)]} N dx = \\ &= \int_{-1}^\infty e^{-N(1+x)+N \log(N(1+x))} N dx = e^{-N} N^{N+1} \int_{-1}^\infty e^{N(\log(1+x)-x)} dx = \\ N \left(\frac{N}{e}\right)^N &\int_{-1}^\infty e^{-N\left(\frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} + \dots\right)} dx = N \left(\frac{N}{e}\right)^N \int e^{-N\frac{x^2}{2}} e^{N\left(\frac{x^3}{3} - \frac{x^4}{4} + \dots\right)} dx. \quad (2) \end{aligned}$$

To go further use identity:

$$\sqrt{2\pi N} e^{-\frac{Nx^2}{2}} = \int e^{\frac{-k^2}{2N}} e^{ikx} dk \quad (***)$$

The meaning of his identity see later (see equation (**Fedoruk identity**))

This identity implies that

$$\begin{aligned} \int e^{-N\frac{x^2}{2}} \varphi(x) dx &= \frac{1}{\sqrt{2\pi N}} \int e^{\frac{-k^2}{2N}} \underbrace{e^{ikx} \varphi(x)}_{2\pi \tilde{\varphi}(-k)} dk dx = \\ &= \frac{2\pi}{\sqrt{2\pi N}} \int e^{\frac{-k^2}{2N}} \tilde{\varphi}(k) dk = \frac{2\pi}{\sqrt{2\pi N}} e^{\frac{1}{2N} \left(\frac{d}{dx}\right)^2} \varphi(x) \Big|_{x=0} \quad (2a) \end{aligned}$$

Identity (***) allows us to rewrite (2) as (2a), i.e. in terms of differential operator. Here we used Fourier transformation:

$$e^{\frac{1}{2N}\left(\frac{d}{dx}\right)^2} \varphi(x) \Big|_{x=0} = \left[e^{\frac{1}{2N}\left(\frac{d}{dx}\right)^2} \int \tilde{\varphi}(k) e^{ikx} \right]_{x=0} =$$

$$\left[\int \tilde{\varphi}(k) \left(e^{-\frac{k^2}{2N}} \right) e^{ikx} \right]_{x=0} = \int \tilde{\varphi}(k) e^{-\frac{k^2}{2N}} dk .$$

(see also e.g. MathBlog on September 2020)

[In fact doing this trick we use the 'Fedoruk' identity

$$\int f(x_0 - x)u(x)dx = \int \tilde{f}(k)e^{ik(x_0-x)}\tilde{u}(p)e^{ipx} dxdkdp =$$

$$\int \tilde{f}(k)e^{ikx_0}\tilde{u}(p)e^{i(p-k)x} dxdkdp = 2\pi \int \tilde{f}(k)e^{ikx_0}\tilde{u}(p)\delta(p-k)dkdp =$$

$$2\pi \int \tilde{f}(k)e^{ikx_0}\tilde{u}(k)dk = 2\pi \tilde{f}\left(\frac{1}{i}\frac{d}{dx}\right)u(x) \Big|_{x=x_0}, \quad (\text{Fedoruk identity})$$

where \tilde{f}, \tilde{u} are Fourier images of functions f, u :

$$g(x) = \int \tilde{g}(k)e^{ikx} dk, \quad \text{with} \quad \tilde{g}(k) = \frac{1}{2\pi} \int \tilde{g}(k)e^{-ikx} dk .$$

]

Namely apply identity (***) to equation (2) using the fact that according to equation (****), Fourier image of function $f(x) = e^{-\frac{Nx^2}{2}}$ is equal to $\tilde{f}(k) = \frac{e^{-\frac{k^2}{2N}}}{\sqrt{2\pi N}}$:

$$N! = N \left(\frac{N}{e}\right)^N \int_{-1}^{\infty} e^{-N\left(\frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} + \dots\right)} dx =$$

$$N \left(\frac{N}{e}\right)^N \int e^{-N\frac{x^2}{2}} e^{N\left(\frac{x^3}{3} - \frac{x^4}{4} + \dots\right)} dx =$$

$$N \left(\frac{N}{e}\right)^N \int \underbrace{e^{-N\frac{x^2}{2}}}_f \underbrace{e^{N\left(\frac{x^3}{3} - \frac{x^4}{4} + \dots\right)}}_u dx =$$

$$N \left(\frac{N}{e}\right)^N \cdot 2\pi \left\{ \frac{e^{\left(-\frac{1}{2N}\left(\frac{1}{i}\frac{d}{dx}\right)^2\right)}}{\sqrt{2\pi N}} \left[e^{N\left(\frac{x^3}{3} - \frac{x^4}{4} + \dots\right)} \right] \right\}_{x=0} =$$

$$\left(\frac{N}{e}\right)^N \cdot \sqrt{2\pi N} e^{\frac{1}{2N}\frac{d^2}{dx^2}} \left[e^{N\left(\frac{x^3}{3} - \frac{x^4}{4} + \dots\right)} \right]_{x=0} =$$

... and here the mystery began: since the expression $\left[e^{N\left(\frac{x^3}{3} - \frac{x^4}{4} + \dots\right)} \right]$ possesses only the terms of the order ≥ 3 in exponent the last integral possess only terms which are proportional to $\frac{1}{N}$:

$$\frac{1}{N} \approx \hbar$$

$$\int e^{-\frac{1}{N}\left(\frac{d}{dx}\right)^2} \left[e^{N\left(\frac{x^3}{3} - \frac{x^4}{4} + \dots\right)} \right] dx = 1 + \frac{1}{12N} + \frac{1}{288N^2} + \dots$$

Try to show it. We have

$$N! = \left(\frac{N}{e}\right)^N \sqrt{2\pi N} \left(1 + O\left(\frac{1}{N}\right)\right),$$

where

$$\begin{aligned} 1 + O\left(\frac{1}{N}\right) &= \left\{ e^{-\frac{1}{2N}\left(\frac{1}{i}\frac{d}{dx}\right)^2} \left[e^{N\left(-\frac{x^3}{6} + \frac{x^4}{24} + \dots\right)} \right] \right\}_{x=0} = \\ &= \left[1 - \frac{1}{2N} \left(\frac{1}{i}\frac{d}{dx}\right)^2 + \frac{1}{8N^2} \left(\frac{1}{i}\frac{d}{dx}\right)^4 - \frac{1}{48N^3} \left(\frac{1}{i}\frac{d}{dx}\right)^6 + \dots \right] \\ &= \left[1 + N \left(\frac{x^3}{3} - \frac{x^4}{4} + \dots\right) + \frac{1}{2}N^2 \left(\frac{x^3}{3} - \frac{x^4}{4} + \dots\right)^2 + \dots \right] = \\ &= \left[1 + \frac{1}{2n} \frac{d^2}{dx^2} + \frac{1}{8n^2} \frac{d^4}{dx^4} + \frac{1}{48n^3} \frac{d^6}{dx^6} + \dots \right] \\ &= \left[1 + n \left(\frac{x^3}{3} - \frac{x^4}{4} + \dots\right) + \frac{1}{2}n^2 \left(\frac{x^3}{3} - \frac{x^4}{4} + \dots\right)^2 + \dots \right] = \end{aligned} \quad (0)$$

Calculate in detail contribution given by this expression.

First show that all terms which give contribution to this expression have negative degree by N .

Consider the action of the terms $\left[\frac{1}{2n} \left(\frac{1}{i}\frac{d}{dx}\right)^2 \right]^\lambda$ which act on the monoms

$$F_{p_1} F_{p_2} \dots F_{p_r},$$

where every F_{p_k} is a monom of the order p_k , ($p_i \geq 3$) which belong to the term

$$n \left(\frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \dots \right).$$

The condition that the action of the operator $\left[\frac{1}{2n} \left(\frac{1}{i} \frac{d}{dx}\right)^2\right]^\lambda$ on the monom $F_{p_1} F_{p_2} \dots F_{p_r}$, is not zero is the following:

$$2\lambda = p_1 + p_2 + \dots + p_r,$$

and the power of n corresponding to this monom is equal to

$$\left(\frac{1}{n}\right)^\lambda \cdot n^r = n^{r-\lambda}.$$

Use this formula.

First of all note that all $p_i \geq 3$, hence $2\lambda \geq 3r$, hence

$$r - \lambda \leq r - \frac{3r}{2} < 0.$$

Thus in the case of Stirling formula we proved hard part of stationary phase method, showing that only negative powers of N (positive powers of $\hbar \approx \frac{1}{N}$) contribute to the integral.

Now calculate the contribution of power $\frac{1}{n^k}$ for $k = 1, 2, 3, \dots$

We have that

$$\begin{cases} 2\lambda = p_1 + \dots + p_r \\ r - \lambda = -k \end{cases}, \quad (p_i \geq 3).$$

Hence

$$\begin{cases} 2\lambda = p_1 + \dots + p_r \geq 3r \\ \lambda = r + k \end{cases} \Rightarrow 2r + 2k \geq 3r, \text{ i.e. } r \leq 2k$$

Thus we see that contribution to terms of order $\frac{1}{n^k}$ is given by action of $\exp -\frac{1}{n} \left(\frac{1}{i} \frac{d}{dx}\right)^2$ on terms which possess not more than $2k$ monoms

Now calculate contribution to the term $\frac{1}{N}$.

I Calculate contribution to $\frac{1}{n}$, $k = 1$:

$$\begin{cases} 2\lambda = p_1 + \dots + p_r \geq 3r \\ \lambda = r + 1 \end{cases} \Rightarrow r = 1, 2.$$

$$a) r = 1, \begin{cases} 2\lambda = p_1 \\ \lambda = r + 1 = 2 \end{cases}, p_1 = 4, \quad b) r = 2, \begin{cases} 2\lambda = p_1 + p_2 \geq 3r \\ \lambda = r + 1 = 3 \end{cases}, p_1 = p_2 = 3,$$

According to these calculations and to equation (contribution), the contribution is equal to:

$$\frac{1}{8N^3} \left(\frac{1}{i} \frac{d}{dx}\right)^4 \left[n \left(-\frac{x^4}{4}\right) \right] + \left(-\frac{1}{48N^3}\right) \left(\frac{1}{i} \frac{d}{dx}\right)^6 \left[\frac{1}{2} N^2 \left(\frac{x^2}{3}\right)^2 \right] = \frac{1}{12N}$$

Thus

$$N! = \left(\frac{N}{e}\right)^N \sqrt{2\pi N} \left(1 + \frac{1}{12N} + O\left(\frac{1}{N^2}\right)\right).$$