Today is 28-st December. This is the special day of the Year.

Barnes functions and integrals

We calculate integral $\int_\delta^\infty \frac{e^{-t}}{t}dt$ and related with it integrals using one analytical continuation method which was inpsired by studying of Barnes functions

In the theory of Barnes functions it is very useful to use the following formula:

$$\log G = \Psi(z|a_1,\dots,a_n) = \Psi(z|a_1,\dots,a_n) = \frac{d}{ds}\zeta(s,z|a_1,\dots,a_n)\big|_{s=0} = \tag{1a}$$

$$= \int_0^\infty \left(A(z,t) - A_-(z,t) - A_0(z)e^{-t}\right)\frac{dt}{t}\,, \tag{1b}$$

where

$$A(z,t) = \frac{e^{-zt}}{\prod_{i=1}^k (1-e^{-a_i t})}\,. \tag{1c}$$

Formula (1a) is just the definition of Barnes function, formula (1b) is very useful interal expression for analytical continuation. We explain here again how we come to formula (1b) but we will consider more general case introducing an additional parameter.

Recall that for an 'arbitrary' function $A(t)$ [1)] one can consider function

$$\zeta_A(s) = \frac{1}{\Gamma(s)}\tilde{A}(s)\,, \quad \text{where Mellin transform } \tilde{A}(s) = \int_0^\infty t^{s-1}A(t)dt\,. \tag{2}$$

A function $\frac{d}{ds}\zeta_A(s)$ is defined by this formula for s which have enough big real part. It can be analyticaly continued to meromorphic function for all s. We will perform it here in a more general way: We have that for Mellin transform

$$\tilde{A}(s) = \int_0^1 t^{s-1}A(t)dt + \int_1^\infty t^{s-1}A(t)dt =$$

$$\int_0^1 t^{s-1}(A(t) - A_-(t) - A_0)dt + \sum_{k<0}\frac{A_k}{k+s} + \frac{A_0}{s} + \int_1^\infty t^{s-1}A(t)dt\,. \tag{3}$$

As usual we denote $A_-(t)$ the Laurent polynomial with negative powers of t, and A_0 the free term:

$$A(t) = \underbrace{\sum_{k<0} A_k t^k}_{A_-(t)} + A_0 + O(t) \tag{4}$$

Now we calculate analytical continuation of function $\frac{d}{ds}\zeta_A(s)$ at the point $s=0$. (The 'naive' integral representation (2) diverges if for example function $A_-(t) \neq 0$ or $A_0 \neq 0$)

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1) We consider functions $A(t)$ which are smooth for positive t, such that for every N $\int t^N A(t)$ converges at infinity and for every function $A(t)$ there exists r such that $t^r A(t)$ is smooth at $t=0$.

Perform analytical continuation.

We have using (3) and (4) that

$$\frac{d}{ds}\zeta_A(s)\big|_{s=0} = \left\{\frac{d}{ds}\left[\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}A(t)dt\right]\right\}_{s=0} =$$

$$\left\{\frac{d}{ds}\left[\frac{1}{\Gamma(s)}\left(\int_0^1 t^{s-1}\left(A(t)-A_-(t)-A_0\right)dt + \int_0^1 t^{s-1}\left(A_-(t)+A_0\right)dt + \int_1^\infty t^{s-1}A(t)dt\right)\right]\right\}_{s=0}$$

$$= \left\{\frac{d}{ds}\left[\frac{1}{\Gamma(s)}\left(\int_0^1 t^{s-1}\left(A(t)-A_-(t)-A_0\right)dt + \sum_{k\leq 0}\frac{A_k}{k+s} + \int_1^\infty t^{s-1}A(t)dt\right)\right]\right\}_{s=0}. \tag{5}$$

Use the fact that $\Gamma(s) \approx \frac{1}{s}$ in a vicinity of 0, hence for an arbitrary function $f(s)$

$$\frac{d}{ds}\left(\frac{f(s)}{\Gamma(s)}\right) = f(0)\,,\ \text{if } f(0) \text{ is well-defined}$$

Thus integral (5) implies that

$$\frac{d}{ds}\zeta_A(s)\big|_{s=0} = \int_0^1 (A(t)-A_-(t)-A_0)\frac{dt}{t} + \sum_{k<0}\frac{A_k}{k} + \frac{d}{ds}\left(\frac{A_0}{s\Gamma(s)}\right)\big|_{s=0} + \int_1^\infty A(t)\frac{dt}{dt} =$$

$$\int_0^1 (A(t)-A_-(t)-A_0)\frac{dt}{t} + \int_1^\infty A(t)\frac{dt}{dt} + \sum_{k<0}\frac{A_k}{k} - A_0\Gamma'(1) =$$

$$\int_0^1 (A(t)-A_-(t)-A_0)\frac{dt}{t} + \int_1^\infty \left(A(t)-A_-(t)\right)\frac{dt}{dt} - A_0\Gamma'(1)\,.$$

Now consider a function φ such that $\varphi(1) = 1$, Transforming last integral we come to

$$\frac{d}{ds}\zeta_A(s)\big|_{s=0} = \int_0^\infty (A(t)-A_-(t)-A_0\varphi(t))\frac{dt}{t} + A_0\left[\int_0^1 (\varphi(t)-1)\frac{dt}{t} + \int_1^\infty \varphi(t)\frac{dt}{t} - \Gamma'(1)\right] =$$

$$\frac{d}{ds}\zeta_A(s)\big|_{s=0} = \int_0^\infty (A(t)-A_-(t)-A_0\varphi(t))\frac{dt}{t} + A_0\left[\int_0^1 (\varphi(t)-1)\frac{dt}{t} + \int_1^\infty \varphi(t)\frac{dt}{t} - \Gamma'(1)\right]. \tag{6}$$

This integral converges if $\varphi(t)$ is rapidly decreasing at infinity and is finite at zero. It defines the value of the function $\frac{d}{ds}\zeta_A(s)$ at the point $s = 0$.

Play with function a φ

We consider two examples $\varphi = e^{-t}$ and $\varphi = e^{-ct}$

First example

$$\varphi = A(t) = e^{-t}$$

In this case $\tilde{A}(s)=\Gamma(s)$, $\zeta_A(s)=1$, $\frac{d}{ds}\zeta_A(s)\big|_{s=0}=0$. We have

$$0=\frac{d}{ds}\zeta_A(s)\big|_{s=0}=\int_0^\infty(A(t)-A_-(t)-A_0\varphi(t))\frac{dt}{t}+A_0\left[\int_0^1(\varphi(t)-1)\frac{dt}{t}+\int_1^\infty\varphi(t)\frac{dt}{t}-\Gamma'(1)\right]$$

$$=\int_0^\infty(e^{-t}-0-e^{-t})\frac{dt}{t}+1\cdot\left[\int_0^1(e^{-t}-1)\frac{dt}{t}+\int_1^\infty e^{-t}\frac{dt}{t}-\Gamma'(1)\right],$$

i.e.

$$\int_0^1\frac{e^{-t}-1}{t}dt+\int_1^\infty\frac{e^{-t}}{t}dt=\Gamma'(1) \qquad \text{Integral} \quad *$$

Note that in this case we come to equation (1b),
Now consider more general case:

Second example

$$\varphi=A(t)=e^{-ct},$$

In this case $\tilde{A}(s)=\int t^{s-1}A(t)dt=\Gamma(s)c^{-s}$, $\zeta_A(s)=c^{-s}$, $\frac{d}{ds}\zeta_A(s)\big|_{s=0}=-\log c$. We have

$$-\log c=\frac{d}{ds}\zeta_A(s)\big|_{s=0}=\int_0^\infty(A(t)-A_-(t)-A_0\varphi(t))\frac{dt}{t}+A_0\left[\int_0^1(\varphi(t)-1)\frac{dt}{t}+\int_1^\infty\varphi(t)\frac{dt}{t}-\Gamma'(1)\right]$$

$$=\int_0^1(e^{-ct}-1)\frac{dt}{t}+\int_1^\infty e^{-ct}\frac{dt}{t}-\Gamma'(1),$$

i.e.

$$\int_0^1\frac{e^{-ct}-1}{t}dt+\int_1^\infty\frac{e^{-ct}}{t}dt=\Gamma'(1)-\log c,$$

Choosing c–real positive number we come to

$$\int_0^1\frac{e^{-ct}-1}{t}dt+\int_1^\infty\frac{e^{-ct}}{t}dt=$$

$$\int_0^c\frac{e^{-t}-1}{t}dt+\int_\delta^\infty\frac{e^{-t}}{t}dt=\Gamma'(1)-\log c, \qquad (\text{Integral} \quad **)$$

Or in :other way:

$$\int_c^\infty\frac{e^{-t}}{t}dt=-\log\delta-\Gamma'(1)+\sum_{k=1}^\infty\frac{(-1)^k}{kk!}\delta^k. \qquad (\text{Integral} \quad ***)$$

I think just using analytical continuation formula (6) for calculation of these integral justifies it. Le jeu en vaut le chandelle!

28 December 2015