Bernoulli numbers, Bernoulli polynomials...

We know that Bernoulli numbers appear everywhere in mathematics. I will consider here two topics: basic formulae for integrals leading to Euler-Maclaurin type formulae and Fourier transformations formulae for calculations of ζ function at even points. In both cases naturally appear the same series of polynomials...

Shortly: Bernoulli Polynomials are polynomials $\{B_n\}$ of the degree n which are convenient for integration by part as well as polynomials $\{x^n\}$. They also are orthogonal to constant function. Bernoulli numbers are values of Bernoulli polynomials in boundary points.

§1 Integral and area of trapezium

Everybody who heard about integral knows that $\int_a^b f(t)dt$ equals approximately to the area of trapezoid with altitude $(b-a)$ and parallel sides equal to the values of the function f at the points a, b:

$$\int_a^b f(t)dt \approx (b-a)\cdot\frac{f(a)+f(b)}{2} \tag{1}$$

Very simple question: How this formula follows from the formula of integration by parts ($\int f(x)dx = xf(x) - \ldots$)? (I was surprised realising that I never asked myself this simple question before.)

Answer: Instead $\int f(x)dx = xf(x) - \int xf'(x)dx$ take $\int f(x)dx = (x+c)f(x) - \int (x+c)f'(x)dx$ putting $x+c$ instead x, where c is an arbitrary constant. Thus we come to

$$\int_a^b f(t)dt = (x+c)f(x)\big|_a^b - \int_a^b (t+c)f'(t)dt \tag{2}$$

Now if we choose $c = -\frac{a+b}{2}$ we come to (1).

$$\int_a^b f(t)dt = \left(x - \frac{a+b}{2}\right)f(x)\big|_a^b - \int_a^b \left(x-\frac{a+b}{2}\right)f'(t)dt = \frac{b-a}{2}(f(a)+f(b)) + \ldots \tag{2a}$$

One can go further performing integration by part. Keeping in mind formula (2a) instead an expansion

$$\int f(x)dx = xf(x) - \frac{x^2}{2}f'(x) + \frac{x^3}{3!}f''(x) - \frac{x^4}{4!}f'''(x) + \ldots \tag{3}$$

we consider an expansion

$$\int f(x)dx = B_1(x)f(x) - \frac{B_2(x)}{2}f'(x) + \frac{B_3(x)}{3!}f''(x) - \frac{B_4(x)}{4!}f'''(x) + \ldots, \tag{3a}$$

where polynomials $\{B_1(x), B_2(x), B_3(x), \ldots\}$ are defined by the relations $\frac{dB_{k+1}(x)}{dx} = kB_k(x)$:

$$B_1(x) = x + c_1,\ B_2(x) = 2\left(\frac{x^2}{2} + c_1x + c_2\right),\ B_3(x) = 6\left(\frac{x^3}{6} + c_1\frac{x^2}{2} + c_2x + c_3\right),$$

$$B_4(x) = 24\left(\frac{x^4}{24} + c_1\frac{x^3}{6} + c_2\frac{x^2}{2} + c_3x + c_4\right),\ \text{and so on}, \tag{3b}$$

where $c_1, c_2, c_3, \ldots$ are an arbitrary constants. We have for an interval (a, b) that

$$\int_a^b f(t)dt = \sum_{n=1}^N (-1)^{n-1}\frac{B_n(x)}{n!}f^{(n-1)}(x)|_a^b + \frac{(-1)^N}{N!}\int_a^b B_N(t)f^{(N)}(t)dt =$$

$$B_1(b)f(b) - B_1(a)f(a) - \frac{B_2(b)f'(b) - B_2(a)f'(a)}{2} + \frac{B_3(b)f''(b) - B_3(a)f''(a)}{6} + \ldots \tag{4}$$

Now encouraged by the trapezoid formula choose $c_1 = -\frac{a+b}{2}$. Then

$$B_2(a) = B_2(b)\,. \tag{5}$$

since $B_1(a) = B_1(b)$ if $c_1 = -\frac{a+b}{2}$. We want to keep the relation (5) for all $B_k(x)$ for $k \geq 2$:

$$B_k(a) = B_k(b) \quad \text{for all } k \geq 2 \tag{5a}$$

In this case the formula (4) becomes:

$$\int_a^b f(t)dt = (b-a)\cdot\frac{f(a)+f(b)}{2} + \sum_{n\geq 2}\frac{(-1)^{n-1}}{n!}B_n(a)\left(f^{(n-1)}(b) - f^{(n-1)}(a)\right) \tag{6}$$

§2 Bernoulli polynomials and numbers

The condition (5a) fixes uniquely all constants $c_2, c_3, \ldots, c_4, \ldots$ in (3). We come to recurrent formula for polynomials $B_n(x)$:

$$B_0(x) \equiv 1, \quad \begin{cases} B_k(x)\colon \ \ \frac{dB_k(x)}{dx} = kB_{k-1}(x) \\ \int_a^b B_k(x)dx = 0, \ \text{i.e.}\, B_{k+1}(a) = B_{k+1}(b) \end{cases} \quad (k = 1, 2, 3, \ldots) \tag{2.1}$$

One can say roughly that polynomials $B_n(x) = x^n + \ldots$ are "deformations" of polynomials x^n suitable for integration by part.

These polynomials are:

$$\begin{gathered} B_0(x) = 1 \\ B_1(x) = x - \frac{a+b}{2} \\ B_2(x) = (x-a)(x-b) + \frac{1}{6}(b-a)^2 \\ B_3(x) = (x-a)^3 - \frac{3}{2}(x-a)^2(b-a) + \frac{1}{2}(x-a)(b-a)^2 \\ B_4(x) = (x-a)^4 - 2(x-a)^3(b-a) + (x-a)^2(b-a)^2 - \frac{1}{30}(b-a)^4 \\ \ldots. \end{gathered} \tag{2.1a}$$

Consider *normalised* polynomials choosing $a = 0, b = 1$:

$$B_0(x) = 1,\ B'_n(x) = nB_{n-1}, \int_0^1 B_n(x)dx = 0, \text{ i.e.}\, B_{n+1}(0) = B_{n+1}(0), n = 1, 2, \ldots :$$

$$\begin{gathered} B_0(x) = 1 \\ B_1(x) = x - \frac{1}{2} \\ B_2(x) = x^2 - x + \frac{1}{6} \\ B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x \\ B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30} \\ B_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{x}{6} \\ B_6(x) = x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{1}{3}x^2 + \frac{1}{42} \\ B_7(x) = x^7 - \frac{7}{2}x^6 + \frac{7}{2}x^5 - \frac{7}{6}x^3 + \frac{1}{6}x \\ \ldots \end{gathered} \tag{2.2}$$

Exercise 1 Show that relation between normalised polynomials $B_n^{[0,1]}$ in (7) and polynomials $B_n^{[a,b]}$ is

$$B_n^{[a,b]}(x) = (a-b)^n B_n^{[0,1]}\left(\frac{x-a}{b-a}\right) \tag{2.3}$$

This formula controls the behaviour of Bernoulli polynomials under changing of a, b.

We define *Bernoulli number* b_n as a value of polynomial (2.2) at the points 0 or 1 (or polynomial (2.1a) divided by a coefficient $(b-a)^n$)

$$b_n = B_n(0) = B_n(1).$$

We have

$$b_0 = 1, b_1 = -\frac{1}{2}, b_2 = \frac{1}{6}, b_3 = 0, b_4 = -\frac{1}{30}, b_5 = 0, b_6 = \frac{1}{42}, b_7 = 0, \dots$$

Interesting observation:

Proposition 1 *Bernoulli numbers b_n are equal to zero if n is an odd number bigger than 1.*

This proposition follows from the following very beautiful property of Bernoulli polynomials:

Proposition 2 *Let $\{B_n(x)\}$ be a set of Bernoulli polynomials corresponding to the interval (a, b) (see eq. (7)). Let P be a reflection with respect to the middle point $\frac{a+b}{2}$ of the interval (a, b):*

$$P\colon\ x \mapsto a + b - x \tag{2.4}$$

Then all Bernoulli polynomials (except B_1) are eigenvectors of this transformation:

$$B_n(Px) = B_n(x) \text{ for all even } n,\ n = 0, 2, 4, \dots \tag{2.5a}$$

and

$$B_n(Px) = -B_n(x) \text{ for all odd } n \geq 3,\ n = 3, 5, 7, \dots \tag{2.5b}$$

Indeed it follows from (2.5b) that $b_n = B_n(a) = -B_n(b) = -b_n$ for odd $n \geq 3$. Thus $b_n = 0$ for $n = 3, 5, 7, \dots$.

The statement of this Proposition 2 is irrelevant to the choice of a, b. To prove the Proposition it is suffice to consider the special case $a = -b$. In this case the transformation P in (2.4) is just $x \mapsto -x$. Thus in this case the statement of Proposition is that Bernoulli polynomials $B_n(x)$ are even polynomials ($B_n(x) = B_n(-x)$) if n is even, and they are odd polynomials if n is an odd number greater than 1 ($B_n(x) = -B_n(-x)$).

Prove it by induction. Suppose that for $n \leq 2N$ this is true. Then consider polynomial $B_{2N}(x)$. We have that $\int_{-a}^{a} B_{2N}(x)dx = 0$, hence $\int_0^a B_{2N}(x)dx = 0$ since by induction hypothesis this is an even polynomial. Hence

$$B_{2N+1}(x) = (2N+1)\int_0^x B_{2N}(t)dt\,.$$

Indeed this polynomial obeys the differential equation $B'_{2N+1}(x) = (2N+1)B_{2N}(x)$ This polynomial is also an odd polynomial. Hence it obeys the boundary condition $\int_{-a}^{a} B_{2N+1}(x)dx = 0$. It remains to prove that B_{2N+2} is an even polynomial. We have that $B_{2N+2}(x) = \int_0^x B_{2N+1}(t)dt + c_{2N+2}$, where c_{2N+2} is a constant chosen by the boundary condition $\int_a^a B_{2N+2}(x)dx = 0$. We see that B_{2N+2} is even since B_{2N+1} is an odd polynomial and constant is an even polynomial.

§3 Integral and area of trapezium (revisited)—Euler-Maclaurin formula

Now equipped by the knowledge of formulae return to the last formula from the first paragraph:

$$\int_a^b f(t)dt = (b-a)\cdot\frac{f(a)+f(b)}{2} + \sum_{n\geq 2}\frac{(-1)^{n-1}}{n!}B_n(a)\left(f^{(n-1)}(b) - f^{(n-1)}(a)\right) = \tag{3.1}$$

$$(b-a)\cdot\frac{f(a)+f(b)}{2} + \sum_{n\geq 2}\frac{(-1)^{n-1}}{n!}b_n(b-a)^n\left(f^{(n-1)}(b) - f^{(n-1)}(a)\right) = \tag{3.1a}$$

$$(b-a)\cdot\frac{f(a)+f(b)}{2} + \sum_{k\geq 1}\frac{(-1)^{2k-1}}{(2k)!}b_{2k}(b-a)^{2k}\left(f^{(2k-1)}(b) - f^{(2k-1)}(a)\right) = \tag{3.1a}$$

$$(b-a)\cdot\frac{f(a)+f(b)}{2} + \frac{1}{6}(b-a)^2\left(f'(b) - f'(a)\right) - \frac{1}{30}(b-a)^4\left(f'''(b) - f'''(a)\right) + \dots$$

We are ready to write down asymptotic formula for series: Dividing the interval $[0,1]$ on $N+1$ parts consider the formula above for any interval $\left[\frac{k}{N}, \frac{k+1}{N}\right]$, then making summation we come to:

$$\int_0^1 f(x)dx = \frac{1}{2N}f(0) + \left(f\left(\frac{1}{N}\right) + f\left(\frac{2}{N}\right) + \ldots + f\left(\frac{N-1}{N}\right)\right) + \frac{1}{2N}f(0) +$$

$$+ \sum_{k\geq 1} \frac{(-1)^{2k-1}}{(2k)!} \frac{b_{2k}}{N^{2k}} \left(f^{(2k-1)}(1) - f^{(2k-1)}(0)\right)$$

it is well-known Euler-Macklourin asymptotic formula.

Remark Our notation for bernoulli numbers is not standard. Bernoulli numbers are $B_n = b_{2n}$.

Exercise Use this formula for the functions $f = x^r$ to express sums $\sum_{i=1}^N i^r$ via Bernoulli numbers.

§4 **Fourier image of Bernoulli polynomials and ζ-function**

Bernoulli polynomials are deformations of x^n which are convenient for integration by part. Function e^x is eigenvalue of derivation operator. This means that Bernoulli polynomials have "good" Fourier transform. Do calculations. Consider Fourier polynomials for the interval $[0,1]$ (see 2.2) and an orthonormal basis $c_k\{e^{2\pi ikx}\}$ where .... Indeed

$$\left\langle B_n(x), e^{2\pi ik} \right\rangle = \int_0^1 B_n(x), e^{2\pi ikx} \sim \frac{1}{k^n}$$

Hence

$$\left\langle B_n(x), e^{2\pi ik} \right\rangle \sim \sum \frac{1}{k^{2n}} = \zeta(2n)$$

Notice that square of the norms of Bernoulli polynomials can be expressed via Bernoulli numbers due to their properties...