Funny integrals (Borwein integrals)

Yesterday my son David showed me the funny integrals on the web-blog of Steven Lundsburg (professor of Economics, Rochester University), cite:

http://www.thebigquestions.com/2012/03/26/loose-ends/.

I enjoyed them so much! Hope you will enjoy too

Consider integrals:

$$A_n = \int_0^\infty \prod_{k=1}^n \frac{\sin \frac{x}{2k-1}}{\frac{x}{2k-1}} dx,$$

i.e.

$$A_1 = \int_0^\infty \frac{\sin x}{x} dx\,, \quad A_2 = \int_0^\infty \frac{\sin x}{x} \frac{\sin \frac{x}{3}}{\frac{x}{3}} dx\,, \quad A_3 = \int_0^\infty \frac{\sin x}{x} \frac{\sin \frac{x}{3}}{\frac{x}{3}} \frac{\sin \frac{x}{5}}{\frac{x}{5}} dx\,, \dots \qquad (1)$$

It really looks surprising but the sequence $\{A_1, A_2, A_3, \dots\}$ looks in the following way:

$$\left\{\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \dots\right\}$$

Yes, the eighth term is not equal to $\frac{\pi}{2}$. As author of the blog claims it is equal to

$$A_8 = \frac{467807924713440738696537864469}{935615849440640907310521750000}\pi \quad (???!!!)$$

This is based on the paper [1].

Shortly the idea of calculating integrals (as it was explained in the article [1]) is based on the remark that Fourier image of the function $\frac{\sin ax}{x}$ is characteristic function of the interval $[-a, a]$:

$$\frac{1}{\sqrt{2}\pi} \int_{-\infty}^\infty e^{ix\xi} \frac{\sin ax}{x} dx = \begin{cases} 1 \text{ if } |\xi| < a \\ 0 \text{ if } |\xi| > a \end{cases}.$$

Now using the fact that Fourier image of product is convolution one can reduce the problem of calculating these integrals to calculating integrals of functions related with convolutions of characteristic functions of intervals.

§ Calculations without Fourier transformation

At the end of the article [1] authors consider another I much more elementary method which is found on the following identity:

$$\int_0^\infty \prod_{k=1}^n \frac{\sin a_k x}{x} \prod_{i=1}^m \cos c_i x \frac{\sin x}{x} dx = \frac{\pi}{2} \prod_{k=1}^n a_k\,. \qquad (2)$$

in the case if

$$\sum |a_k| + \sum |c_i| \leq 1. \qquad (2a)$$

This identity goes to the work [2] of C. Stormer in 1885 (see the detailes in [1].)

This identity implies why the first seven terms in (1) are equal to $\frac{\pi}{2}$. Indeed one may read the identity in the following way: for positive integers $a_1, a_2, \ldots, a_k$

$$\int_0^\infty \prod_{k=1}^n \frac{\sin a_k x}{a_k x} \frac{\sin x}{x} dx = \frac{\pi}{2} \text{ in the case if } \sum a_k \leq 1 \tag{3}$$

Now we see that integral for (1)

$$A_n = \int_0^\infty \prod_{k=1}^n \frac{\sin \frac{x}{2k-1}}{\frac{x}{2k-1}} dx = \frac{\pi}{2} \tag{4}$$

$$\text{if } \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1} \leq 1 \tag{4a}$$

This is the case for first seven integrals in (1), but this is not the case for the integral A_8. Indeed

$$\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} =$$

$$\left(\frac{1}{3} + \frac{1}{5} + \frac{1}{9}\right) + \left(\frac{1}{7} + \frac{1}{11} + \frac{1}{13}\right) = \frac{15+9+5}{45} + \frac{143+77+91}{1001} = \frac{29}{45} + \frac{311}{1001} = \frac{43024}{45045} < 1$$

but $\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{15} =$

$$\frac{43024}{45045} + \frac{1}{15} = \frac{43024 + 3003}{45045} = \frac{46027}{45045} > 1.$$

Nevertheless we still can calculate A_8 an come to monstrous answer above using integrals below.

A proof of indentity (2) and calculations below are founded on the following basic identities*:

$$\frac{\sin a_k x}{x} \cos c_i x = \frac{1}{2} \frac{\sin(a_k + c_i)x}{x} + \frac{1}{2} \frac{\sin(a_k - c_i)x}{x},$$

and

$$\frac{\sin a_k x}{x} = \int_0^{a_k} \cos tx dt.$$

It follows from these identities that the integrand in relation (4) can be represented by the following integral:

$$\prod_{k=1}^n \frac{\sin a_k x}{x} \prod_{i=1}^m \cos c_i x \frac{\sin x}{x} =$$

* See [1] for details.The proof below is a variation on...

$$\int \cdots \int_{\substack{0 \le t_i \le a_i, \\ (i=1,\dots,k)}} \frac{\sum_{\pm} \sin(x \pm t_1 x \pm \ldots \pm t_n x \pm c_1 x \pm \ldots + c_m x)}{2^{n+m}} dt_1 \ldots dt_n \,. \qquad (5)$$

(Here the summation goes over all combinations of sings $\pm$.)

Hence due to the fact that

$$\int_0^\infty \frac{\sin ax}{x} dx = \begin{cases} 1 \text{ if } a > 0 \\ -1 \text{ if } a < 0 \end{cases} .$$

we see that if condition (2a) is obeyed then the integral (2) equals to the integral of constant $\frac{\pi}{2}$ over rectangular polyhedron with volume $a_1 \ldots a_m$. This implies the identity (2).

For example calculate $\int_0^\infty \frac{\sin ax}{x} \frac{\sin x}{x} dx$. We have that

$$\int_0^\infty \cos ax \frac{\sin x}{x} dx = \frac{1}{2} \int_0^\infty \frac{\sin(1+a)x + \sin(1-a)x}{x} dx = \frac{\pi}{2} \text{ if } |a| \le 1.$$

Thus

$$\int_0^\infty \frac{\sin ax}{x} \frac{\sin x}{x} dx = \int_0^a dt \int \int_0^\infty \cos tx \frac{\sin x}{x} dx = \int_0^a dt \frac{\pi}{2} dx = a \frac{\pi}{2} \,.$$

We see that $\int_0^\infty \frac{\sin ax}{ax} \frac{\sin x}{x} dx = \frac{\pi}{2}$ if $|a| \le 1$ (It is equal to $\frac{\pi}{2a}$) for $|a| \ge 1$.

In the similar way we may calculate $\int_0^\infty \frac{\sin a_1 x \sin a_2 x \sin x}{x^3} dx$ for positive a_1, a_2 such that $a_1 + a_2 \le 1$

$$\int_0^\infty \frac{\sin a_1 x \sin a_2 x \sin x}{x^3} dx = \int \in \int_{\substack{0 \le t_1 \le a_1 \\ 0 \le t_2 \le a_2 \\ 0 \le x \le \infty}} \left(\frac{\sum_{\pm} sin(x \pm t_1 x \pm t_2 x)}{8} \right) dt_1 dt_2 dx \,.$$

The integral of integrand over x equals to $\frac{\pi}{2}$ since $1 \pm t_1 \pm t_1 \ge 0$. Hence integral equals to $a_1 a_2 \frac{\pi}{2}$.

In the case if condition (2a) is not obeyed we still may use these methods.

Remark When I wrote this text I knew that in Wikipedia these integrals are called Borwein integrals.....

References

[1] David Borwein and Jonathan Borwein. *Some remarkable properties of sinc and related integrals.* The Ramanujan Journal 5(2001), no, 1, pp.73—89

See also http://www.thebigquestions.com/borweinintegrals.pdf. (1991)

[2a]C. Stormer, *Sur generalisation de la formulae* $\frac{\phi}{2} = \frac{\sin\phi}{1} - \frac{\sin\phi}{2} + \frac{\sin 3\phi}{3} - \ldots$, Acta Math. **19** (1885), pp. 341-350

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