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Two formulae for Lie groups

I will recall here calculations of

$$\Sigma_1 = e^{-A} X e^{A} = Ad_A(X) \tag{1}$$

and calculation of

$$\Sigma_2 = e^{-A} e^{A+\varepsilon X}\,, \qquad \text{for } \varepsilon^2 = 0 \tag{2}$$

Both formulae are very useful. (E.g. the second one is inevitable if we would like to perform explicit calculations with left invariant vetor fields.) This is standard to calculate these sums using differential equations. Here I will calculate just by brute force combinatorial calculations.

The first formula comes naturally from differential equation: to calculate Σ_1 in (1) consider a function

$$S_X(t) = e^{-tA} X e^{tA}\,.$$

This function is a solution of differential equation

$$\begin{cases} \frac{dS}{dt} = -[A, S(t)] = -ad_A S(t) \\ S(t)_{t=0} = X \end{cases}$$

Solving this equation we have that

$$S(t) = X - \int_0^t [A, S(\tau)] d\tau\,.$$

Writing this integral recurcsively we come to perturbation expansion or to

$$S(t) = e^{-t ad_A} X\,.$$

Now combinatorial (`brute force`) solution:

$$\Sigma_1 = e^{-A} X e^{A} = \sum_{n,m} \frac{(-1)^n A^n X A^m}{n! m!} = \sum_n \left(\sum_{k=0}^{n} \frac{(-1)^k A^k X A^{n-k}}{k!(n-k)!} \right) =$$

$$\sum_n \frac{1}{n!} \left(\sum_{k=0}^{n} (-1)^k C_n^k A^k X A^{n-k} \right)\,.$$

One can see that polynomials

$$\mathcal{D}_n(A, X) = \sum_{k=0}^{n} (-1)^k C_n^k A^n X A^{n-k}$$

belong to Lee algebra $\mathcal{G}(A, X)$, (they are up to a sign so called Dynkin polynomials):

$$D_n(A, X) = XA^n - nAXA^{n-1} + \ldots + (-1)^n A^n X =$$

$$(-1)^n [A, \ldots, [A, X \underbrace{] \ldots]}_{n\text{times}} = ad_A^n X\,. \qquad (Dynkin)$$

$$\mathcal{D}_1 = -AX + XA = -[A, X]\,, \mathcal{D}_2 = XA^2 - 2AXA + A^2X = [A, [A, X]]$$

$$\mathcal{D}_3 = XA^3 - 3AXA^2 + 3A^2XA - A^3X = -[A, [A, [A, X]]] \quad ,$$

and so on.

We come to

$$\Sigma_1 = \sum_n \frac{1}{n!} \left(\sum_{k=0}^{n} (-1)^k C_n^k A^k X A^{n-k} \right) = \sum_n \frac{(-1)^n ad_A^n X}{n!} = e^{-ad_A} X =$$

$$1 - [A, X] + \frac{[A, [A, X]]}{2} - \frac{[A, [A, [A, X]]]}{6} + \ldots$$

Much more funny formula (2):

Differential equation. Consider function

$$S(t) = e^{-tA} e^{t(A+\varepsilon X)}$$

it obeys equation:

$$\begin{cases} \frac{dS}{dt} = e^{-tA} \varepsilon X e^{tA} S(t) \\ S(t)_{t=0} = X \end{cases}$$

Brute force calculations:

$$\Sigma_2 = e^{-A} e^{A+\varepsilon X} =$$

$$\sum \frac{(-1)^n A^n}{n!} \left(\sum \frac{A^m}{m!} + \sum \frac{\overbrace{A^{m-1}X + A^{m-2}XA + A^{m-3}XA^2 + \ldots + XA^{m-1}}^{m \text{ terms}}}{m!} \right) =$$

$$1 + \sum_r \left(\sum_{p+q=r} \frac{(-1)^p A^p}{p!} \frac{\overbrace{A^{r-q-1}X + A^{r-q-2}XA + A^{r-q-3}XA^2 + \ldots + XA^{r-q-1}}^{(r-q) \text{ terms}}}{(r-q)!} \right) =$$

$$1 + \sum_{m,n} \left(\sum_p \frac{(-1)^p A^p}{p!} \right) \frac{A^m X A^n}{(m+n+1)!}$$

Here changing $p+m \to m$ we come to

$$\Sigma_2 = 1 + \sum_{m,n,p=0,\dots,m} \frac{(-1)^p A^m X A^n}{p!((m-p)+n+1)!} = \sum_{m,n} t_{mn} \frac{A^m X A^n}{(m+n+1)!}\,,$$

where

$$t_{mn} = \sum_{p=0}^{m} \frac{(-1)^p}{p!(m+n+1-p)!}\,.$$

Observation

$$t_{mn} = (-1)^m C_{m+n}^m\,.$$

It follows from this observation that from formula (*Dynkin*) for Dynkin polynomials, that

$$\Sigma_2 = 1 + \sum_{m,n} t_{mn} \frac{A^m X A^n}{(m+n+1)!} = 1 + \sum_N \frac{1}{N+1} \underbrace{\left(\sum_{k=0}^{N} (-1)^k C_N^k A^k X A^{N-k}\right)}_{\text{Dynkin's polynomial } \mathcal{D}_N} =$$

$$1 + \sum_{N=0}^{\infty} \frac{(-1)^N ad_A^N X}{(N+1)!} =$$

$$1 + X - \frac{1}{2}[A,X] + \frac{1}{6}[A,[A,X]] - \frac{1}{24}[A,[A,[A,X]]] - \frac{1}{120}[A,[A,[A,[A,X]]]] + \ldots = \,.$$

It remains to prove `Observation`.

`Proof` We will use Pascal's tree identity:

$$C_n^k + C_n^{k+1} = C_{n+1}^{k+1}\,.$$

Then

$$t_{mn} = \sum_{p=0}^{m} \frac{(-1)^p}{p!(m+n+1-p)!} =$$

$$C_{m+n+1}^0 - C_{m+n+1}^1 + C_{m+n+1}^2 - C_{m+n+1}^3 + \ldots + (-1)^m C_{m+n+1}^m =$$

$$[C_{m+n}^0] - [C_{m+n}^0 + C_{m+n}^1] + [C_{m+n}^1 + C_{m+n}^2] - [C_{m+n}^2 + C_{m+n}^3] + \ldots + (-1)^m [C_{m+n}^{m-1} + C_{m+n}^m] =$$

$$= (-1)^m C_{m+n}^m \blacksquare$$