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On one linear algebra problem

Two weeks ago, David (my son) suggested me the following problem: There are 101 coins. For every coin you can divide the rest of the coins on two sets on 50 coins such that these two sets have the same weight. Prove that all the coins have the same weight. Occasionally I came to the solution, it was highly unexpected (I used one combinatorial result) and very beautiful. I totally changed my opinion about this problem (at first I consider it as a boring exercise in linear algebra). I would like to add also, that David was informed about this problem from mathematician, Vladimir Dotsenko, and I never forget the beautiful proof of Nullstelenzats which was suggested by Vladimir (see e.g. the etude "On a simple proof of Nullstelensatz" in my homepage "www.maths.manchester.ac.uk/khudian": (the subsection /Etudes/Algebra/)).

I what follows, there is my solution of this problem. In Appendix 1, I put the solution of this problem suggested by James Montaldi, and in Appendix 2 some statement related with my proof.

Here is my solution:

Denote the weights of the coins by $\{x_1, x_2, \dots, x_{101}\}$. The problem evidently is reduced to the following: Consider the matrix of the size 101×101, $M = ||M_{ik}||$, $i, k = 1, \dots, 101)$, such that all diagonal elements of this matrix are equal to zero and in every row some 50 non-diagonal entries are equal to $+1$ and another fifty non-diagonal entries are equal to -1.

Then it is evident that the vector

$$\mathbf{e} = \{1, 1, \dots, 1\} \tag{1}$$

is the eigenvector of the matrix M with eigenvalue 0. We have to prove that this is all: the space of zero eigenvectors is 1-dimensional, or in other words, the vector $\mathbf{e}$ says that rank of the matirx M is less than 101. We have to prove that it is just equal to 100.

Consider the characterisitic polynomial

$$P(\lambda) = \det(\lambda - M) = \sum_{k=0}^{101} a_k \lambda^{101-k} .$$

We have that $a_0 = 1$, the existence of eigenvector with zero eigenvalue meams that $a_{101} = \det M = 0$. To prove that the zero eigenvalue subspace is one-dimensional, i.e. all the coins have the same weihgt, we have to prove that

$$a_{100} \neq 0 .$$

Now write the formula for a_{100}:

$$a_{100} = \sum \sigma_{k_1 k_2 \ldots k_{101}} M_{1k_1} M_{2k_2} \ldots M_{100k_{100}} M_{101k_{101}} \,, \tag{3}$$

where $\sigma_{k_1 k_2 \ldots k_{101}}$ is the sign of permutation, and in every monom for all $\{i\}$ except just one i_0, $k_i \neq i$, and $M_{ik_i} = \pm 1$. The number of these monoms is equal to $101 \times S_{100}$, where S_{100} is the number of permutations of 100 numbers such that all the elements change the place, and the point is that number S_{100} is ODD!!!! Hence the number of monoms in the expression (3) is odd, and all the monoms are equal to ± 1, i.e. $a_{100} \neq 0$. In other words if we will write the expression for a_{100} over the field $Z_2 = Z \backslash 2Z$ we come to

$$a_{100} = \sum_{i=1}^{101} \left(\sum_{r \neq i, k_r \neq r} M_{rk_r} \right) = 101 \cdot S_{100} = 1 \cdot 1 = 1 \neq 0 (mod 2) \,.$$

The central point of the proof is the fact that S_{100} is an odd number. In fact one can see that the number S_k of permutations of k elements which displace all the elements is odd if k is even, and vice versa, S_k is even if k is odd; see also the etude "On number of permutations wich displace all the elements" in my homepage "www.maths.manchester.ac.uk/khudian": (the subsection /Etudes/Arithmetics))

Appendix 1

The solution above indicates that the field $Z_2 = Z \backslash 2Z$ is the crucial. James Montaldi presented the solution which is just based on the concept of this field. His solution is the following:

for the matrix M which has to be considered (it vanishes on diagonal and it is equal to $+1$ or -1 at all non-diagonal entries) one can just immediately calculate all eigenvectors and eigenvalues, but over the filed Z_2! Indeed consider the matrix $A = M + I$, where I is identity matrix. The A is the matrix such that all the entries of this matrix are equal to $+1$ or -1. The vector $e = 1, 1, 1, \ldots, 1$ which corresponds to the coins of the same weights is the eigenvector of the matrix A with eigenvalue n, and ANY VECTOR WHICH IS ORTHOGONAL TO THE VECTOR e is the eigenvector of A with eigenvalue 0, OVER FIELD Z_2. Thus we see that the matrix $M = A - I$ has one eigenvector with eigenvalue $n-1$ and $n-1$ vectors with eigenvalue 0 over the field Z_2. Hence this matrix M has rank $n-1$ over the field Z_2. Thus for matrix M if $Mx = 0$ then x is proportional to e, i.e. rank of M is equal to $n-1$ if n is odd.

I think that the main step is to change the field $R \to Z_2$. James works just over field Z_2 and his solution is of course more natural. I could not overcome the psichological barriers, and my solution is in terms of even and odd numbers. In fact we have be surprised that the problem about real numbers, the field Z_2 plays the major role in *!

* comment of Yuri Bazlov

Appendix 2

Of course one can prove using this fact more general statement: Let M be $n \times n$ matrix, such that alldiagonal entries vanish, and all non-diagonal entries are equal to ± 1. Then for the characteristic polynomial $\det(\lambda - M) = a_k \lambda^{n-k}$, for every coefficient a_k,

$$a_k = C_n^k S_k = \frac{n!}{k!(n-k)!}(k+1)\,,(mod 2)$$

and in particular

$$a_r k K \neq 0\,, \text{if } C_n^k \text{ and } k+1 \text{ are odd numbers.}$$