An odd number of coins

Let $x_1, \dots, x_{2n+1}$ be a collection of real numbers such that $x_p \neq x_q$ for at least one pair of indices (p, q). We want to prove that there is an index $k, 1 \leq k \leq 2n+1$, for which the set of indices

$$I = \{1, \dots, 2n+1\} \setminus \{k\}$$

cannot be partitioned into two disjoint sets A and B, $I = A \sqcup B$, such that $|A| = |B| = n$ and

$$\sum_{i \in A} x_i = \sum_{i \in B} x_i.$$

First we want to reduce this problem to the case when the numbers $x_i, 1 \leq i \leq 2n+1$, are rational. Consider the $\mathbb{Q}$-linear subspace V of $\mathbb{R}$ generated by the numbers $x_i, 1 \leq i \leq 2n+1$. Suppose that $\{e_1, \dots, e_m\}$ is a basis in V. Then there is a matrix $\{a_{ij}\}, 1 \leq i \leq 2n+1, 1 \leq j \leq m$, with rational entries such that

$$x_i = \sum_{j=1}^{m} a_{ij} e_j$$

for all i. Suppose that $x_p \neq x_q$ for some indices p, q. Then there exists an index r such that $a_{pr} \neq a_{qr}$. Replace the numbers $x_1, \dots, x_{2n+1}$ with the rational numbers $a_{1\,r}, \dots, a_{2n+1\,r}$.

Thus, we can assume that the numbers $x_1, \dots, x_{2n+1}$ are rational. Write these numbers in 2-adic form. There exists a possibly negative integer d such that, for all i,

$$x_i = \sum_{j=d}^{\infty} b_{ij} 2^j,$$

where $b_{ij} \in \{0, 1\}$. Now let s be the smallest integer satisfying the condition that not all numbers $b_{1\,s}, \dots, b_{2n+1\,s}$ are equal to each other. If the number of ones in the collection $b_{1\,s}, \dots, b_{2n+1\,s}$ is odd, choose the index k such that $b_{ks} = 0$. If the number of ones in the collection $b_{1\,s}, \dots, b_{2n+1\,s}$ is even, choose the index k such that $b_{k\,s} = 1$. Then the index k satisfies the required condition.

To avoid dealing with 2-adic numbers, observe that all numbers $x_1, \dots, x_{2n+1}$ can be simultaneously multiplied by a nonzero number or the same number can be added to all $x_1, \dots, x_{2n+1}$. The modified numbers retain the same partition properties. We can multiply the rational numbers $x_1, \dots, x_{2n+1}$ by a nonzero integer to eliminate all denominators and then add to all of them an integer to make them positive. Then we apply the following algorithm to these positive integers. If all these numbers are odd, subtract one from all of them. If they all are even, divide all of them by 2. Eventually some of these numbers will be even and some odd. If the number of odd numbers is even, we remove an odd number. If it is odd, we remove an even number. In the remaining $2n$ numbers there will be an odd number of odd numbers. It is impossible to split them into two groups of n numbers with equal sums.