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On calculation of one determinant

Let $A = ||a_{ik}||$ be $n \times n$ matrix. Calculate determinant of the $n+1 \times n+1$ matrix

$$\begin{pmatrix} a_{ik} & u_k \\ v_i & 0 \end{pmatrix}$$

Try to use the Frobenius formula:

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A - BD^{-1}C) \det D\,.$$

It does not work straightforwardly since $D = 0$. We will use it in the following way:

$$\det \begin{pmatrix} a_{ik} & u_k \\ v_i & 0 \end{pmatrix} = \lim_{\varepsilon \to 0} \left[\det \begin{pmatrix} a_{ik} & u_k \\ v_i & \varepsilon \end{pmatrix} \right] = \lim_{\varepsilon \to 0} \left[\det \left(a_{ik} - \frac{u_i v_k}{\varepsilon} \right) \varepsilon \right] =$$

$$\lim_{\varepsilon \to 0} \left[\det A \det \left(1 - \frac{a^{im} u_m v_k}{\varepsilon} \right) \varepsilon \right] = \lim_{\varepsilon \to 0} \left[\det A \left(1 + \sum_{k \geq 1} \frac{L_k}{\varepsilon^k} \right) \varepsilon \right] =$$

$$\lim_{\varepsilon \to 0} \left[\det A \left[1 - \mathrm{Tr} \left(\frac{a^{im} u_m v_k}{\varepsilon} \right) + \sum_{k \geq 2} \frac{L_k}{\varepsilon^k} \right] \varepsilon \right] =$$

$$\lim_{\varepsilon \to 0} \left(\det A \varepsilon - \det A\, a^{im} u_m v_i + \det A \sum_{k \geq 2} \frac{L_k}{\varepsilon^{k-1}} \right) = -\det A a^{im} u_m v_i\,,$$

since the limit exists. Here $A^{-1} = ||a^{im}||$ is the matrix inverse to the matrix $A = ||a_{ik}||$. We see that in particular all $L_k = 0$ for $k \geq 2$[1)]. Hence

Note that $\det A\, a^{im} = \tilde{a}_{im}$ is cofactor of matrix $||a_{ik}||$, all elements of this matrix are polynomials on entries of matrix $||a_{ik}||$.

Geometrical meaning Let $A = ||a_{ik}||$ be symmetric $n \times n$ matrix. Consider quadric C_A: $a_i k x^i x^k = 0$ in P^{n-1}. One can see that the equation

$$\det \begin{pmatrix} a_{ik} & u_k \\ u_i & 0 \end{pmatrix} = 0$$

defines the pencil C^* of lines $u_m x^m = 0$ which are tangent to quadric C.

Thus we see that pencil C^* dual to the quadric C is defined by the matrix which is cofactor of matrix A.

[1)] In fact $L_k = \mathrm{Tr} \wedge^k K$, where $K^i_m = a^{im} u_m v_k$, and it can be shown straightforwardly that all L_k vanish for $k \geq 2$. The trick is that we do not need to do it.