Irrational powers of integers.

Few days ago Alex Wilkie asked me a question: Let α be a positive real number such that for all integer n, n^α is integer too. Prove that α is an integer too. I came to very beautiful solutions of this problem. Here it is:

Consider "polynomials"

$$P_1(x) = (x+1)^\alpha - x^\alpha\,,$$

$$P_2(x) = (x+2)^\alpha - 2(x+1)^\alpha + x^\alpha\,,$$

$$P_3(x) = (x+3)^\alpha - 3(x+2)^\alpha + 3(x+1)^\alpha - x^\alpha\,,$$

$$P_4(x) = (x+4)^\alpha - 4(x+3)^\alpha + 6(x+2)^\alpha - 4(x+3)^\alpha + x^\alpha$$

and so on...(In fact these polynomials are finite versions of k-th derivative of function x^α.)

One can show that in Taylor series expansion of "polynomial" $P_k(x)$ (with respect to x: $(x+a)^\alpha = x^\alpha + \frac{\alpha(\alpha-1)}{2}x^{\alpha-1}\ldots$ the first k terms disappear:

$$P_k(x) = ...x^{\alpha-k} + \ldots$$

The proof of this fact is simple but it contains a beautiful combinatorics.

This becomes crucial for solution.

Let α be real number such that $\alpha \le k$. Suppose that n^α is integer for all α. If $\alpha < k$ then $P_k(x)$ tends to 0 if x tends to infinity. On the other hand $P_k(x)$ takes integer values at all integer x. Hence we come to conclusion that $P_k(x)$ equals identically to zero if $\alpha < k$. Hence in Taylor series expansion all coefficients are equal to zero. Hence α is integer ∎This is wonderfull is not it? (19.02.2012)

PS *In fact as Alex Wilkie told me the following:*

1) real positive α is integer if $2^\alpha, 3^\alpha$ and 5^α are integers. This is really difficult to prove. There is no an counterexample: such a positive real non-integer α such that m^α, n^α are integer.