One method of Calculating Integral $\int_1^\infty \frac{\exp(-at)}{t}dt$ for small a and Seeley Formulae

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We consider calculating of the integral $\int_1^\infty \frac{\exp(-at)}{t}dt$ using ideas inspired by Seeley formulae. First the calculation. For estimating the behaviour of $\int_1^\infty \frac{\exp(-at)}{t}dt$ for large a one can consider integration by parts

$$\int_1^\infty \frac{\exp(-at)}{t}dt = \frac{e^{-a}}{a}\sum_{n=0}^{N} \frac{n!(-1)^n}{a^n} + \frac{(-1)^N N!}{a^{N+1}}\int_1^\infty e^{-at}t^{N+1} \tag{1}$$

Remark One can see that this formula does not give convergent power series. On the other hand for every fixed N the reminder term is $\frac{e^{-a}o(1}{a^N)}$.

For small a the Eq.(1) is helpless.

We consdider the trivial identity

$$a^{-s} = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\exp(-at)dt \quad \text{for} \quad s > 0 \tag{2}$$

One can rewrite the l.h.s. of (2) in the way to consider its analytical continuation for neibourhood of $s = 0$

$$a^{-s} = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\exp(-at)dt =$$

$$\frac{1}{\Gamma(s)}\left(\int_0^1 t^{s-1}\left(\exp(-at)-1\right)dt + \int_0^1 t^{s-1}dt + \int_1^\infty t^{s-1}\exp(-at)dt\right)$$

$$= \frac{1}{\Gamma(s)}\left(B(s) + \frac{1}{s}\right) \tag{3}$$

where we denote

$$B(s) = \left(\int_0^1 t^{s-1}\left(\exp(-at)-1\right)dt + \int_1^\infty t^{s-1}\exp(-at)dt\right)$$

Hence

$$a^{-s} = \frac{1}{\Gamma(s)}\left(B(s) + \frac{1}{\Gamma(s+1)}\right) \tag{4}$$

The both parts of (3) are defined and can be differentiate in the neibourhood of $s = 0$. Using that

$$\frac{1}{\Gamma(s)}\Big|_{s=0} = 0 \quad \text{and} \quad \left(\frac{d}{ds}\frac{1}{\Gamma(s)}\right)\Big|_{s=0} = \left(\frac{d}{ds}\left(\frac{s}{\Gamma(s+1)}\right)\right)\Big|_{s=0} = 1 \tag{5}$$

we come from (4) to

$$\frac{d}{ds}a^{-s}\Big|_{s=0} = -\log a = B(0) - \Gamma'(1) \tag{6}$$

the derivative $\Gamma'(1)$ is the famous Euler constant

$$\Gamma'(1) = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} - \log n\right) \tag{7}$$

We see from (2—6) that

$$B(0) = \int_0^1 \frac{\exp(-at) - 1}{t}dt + \int_1^\infty \frac{\exp(-at)}{t}dt = \Gamma'(1) - \log a \tag{8}$$

This gives representation for $\int_1^\infty \frac{\exp(-at)}{t}dt$ which is convenient for small a

$$\int_1^\infty \frac{\exp(-at)}{t}dt = \Gamma'(1) - \log a - \int_0^1 \frac{\exp(-at) - 1}{t}dt =$$

$$\Gamma'(1) - \log a - \sum_{n=1}^\infty \frac{(-1)^n a^n}{n!n} = \tag{9}$$

$$-\log a + \Gamma'(1) + a - \frac{a^2}{2\cdot 2!} + \frac{a^3}{3\cdot 3!} - \frac{a^4}{4\cdot 4!} + \ldots$$

(where $\Gamma'(1)$ is nothing but Euler constant C:
$\Gamma'(1) = C = \lim_{n\to\infty}(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \log n)$) Empirically $B(0)$ in (8) can be considered as renormalized value of the integral

$$\int_1^\infty t^{s-1}\exp(-at)dt = \ldots \tag{10}$$

And naively (which happens often in Quantum Field Theory) one can calculate it using Frullany formula:

$$\int \frac{f(at) - f(bt)}{t}dt = f(0)\cdot\log\frac{b}{a}$$

The exact result differs from naive expectation on the Euler constant !!!.
In fact all this stuff is the reduction of the Seeley formulae

I think this very trivial exercise is useful
shtoby vvesti populjarno ideologiju perenormirovok v formulakh Sili (Seleey)

Remark 2

Denote $I(a) = \int_1^\infty \frac{\exp(-at)}{t}dt$ and $K(a) = \sum_{n=1}^\infty \frac{(-1)^n a^n}{n!n}$

We note that

$$\Gamma'(1) = \int_0^\infty \log t e^{-t} dt = \int_0^1 \log t e^{-1} dt + \int_1^\infty \log t e^{-1} dt = K(1) + I(1)$$

hence we come to the identity at $a = 1$. Helas our formula does not give series for $a = 1$...

Remark 3 Tut vsjo zatsepleno: Naprimer

$$\int_0^1 \frac{e^{-at} - 1}{at} dt = \int_0^1 \log t e^{-at} dt$$

The second integral is related with $\Gamma'(1)$.