On number of real roots. (Silvester Theorem)

Theorem The number of real roots of the polynomial $f = x^n + a_{n-1} + \ldots + a_0$ with real coefficients $a_0, \ldots, a_{n-1}$ is equal to the signature of the quadratic form given by the $n \times n$ matrix $a_{ij} = s_{i+j-2}$ where $i, j = 1, \ldots, n$ and $s_k = x_1^k + \ldots + x^k$ are Newton polynomials polynomials on coefficeints ($s_1 = -a_{n-1}$, $s_2 = a_{n-1}^2 - 2a_{n-2}, \ldots$) ($\{x_1, \ldots, x_n\}$ is the set of complex roots of this polynomials.)

I know the very beautiful proof of this Theorem. (It comes from Prasolov + ...)

Assume that all roots are distinct. Consider the set of polynomials $\{h_i(x)\}$ of degree $\leq n-1$ such that polynomial h_i is equal to 1 at the root x_i and it is equal to zero at all other roots,which are equal to 1:

$$h_i(x) = \frac{f(x)}{(x - x_i)f'(x_i)}$$

(In general roots are complex and these polynomials are complex) Note that polynomials $\{h_i(x)\}$ are the base of Lagrange interpolation formula:

For every polynomial p of degree $\leq n$

$$p(x) \equiv p(x_i)h_i(x)$$

These formulae play the role of China reiminders isomorphism on the ring of polynomials)

Now consider complex n-dimensional vector space V of all complex polynomials factorised by f. The set of polynomials $h_i(x)$ is the basis in this space. The components $(p_1, \ldots, p_n)$ of every polynomial with respect of this basis is just the values of these polynomials at roots: $p_i = p(x_i)$ (according Lagrange interpolaion formula)

Every element of this space defines linear operator $L_g : L_g p = gp$ (modulo f)

Consider the symmetric bilinear form A such that its value on every pair g, r is equal to the trace of the of the operator L_{gr}. It is evident that basis $\{h_i\}$ is orthonormal basis with respect of this form because $h_i h_j \equiv 0$ if $i \neq j$:

$$A(h_i, h_j) = \delta_{ij}$$

Hence we come to the formula

$$A(g, r) = g_1 r_1 + \ldots + g_n r_n = \sum_{i=1}^{n} g(x_i) r(x_i)$$

It follows from this formula that

$$A(x^p, x^q) = \sum_{i=1} x_i^{p+q} = s_{p+q}$$

We see that matrix $a_{ij} = s_{i+j-2}$ is just the matrix of symmetri bilinear form A in the real basis $\{1, x, x^2, \ldots\}$.

Now suppose that $2q$ roots of this polynomial are complex and the rest $r - 2q$ are real. Thus $2q$ polynomials $h_1, \ldots h_{2q}$ are complex and the rest are real.

Consider the real basis $\{a_1, b_1, \ldots, a_q, b_q, h_{q+1, \ldots, h_n}\}$, where $h_1 = a_1 + ib_1$, $h_2 = a_1 - ib_1$, $h_3 = a_2 + ib_2$, $h_4 = a_2 - ib_2, \ldots$. Since $A(h_i, h_j) = \delta_{ij}$ hence

$$A(a_i, a_j) = \frac{1}{2}\delta_{ij}, \quad A(a_i, b_j) = 0, \quad A(b_i, b_j) = \frac{-1}{2}\delta_{ij}$$

Hence the signature of this form is equal to $n - 2q$. It is just equal to the number of real roots.