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On Taylor identity

A standard proof of Taylor Theorem for smooth (infinitely differentiable) function,

$$f(x) = \sum_{k=0}^{n} f^{(k)}(x_0) \frac{(x - x_0)^k}{k!)} + O(x^{n+1}) \tag{1}$$

contains a 'nasty' part related with estimation of residul term $O(x^{n+1})$. There is an elegant proof of Taylor Theorem which is based on the identity

$$f(x) = \sum_{k=0}^{n} f^{(k)}(x_0) \frac{((x - x_0)^k}{k!)} + \frac{1}{n!} \int_{x_0}^{x} \frac{d^{n+1}}{dt^{n+1}} f(t)(x - t)^n dt \tag{2}$$

This identity immediately leads to (1).

Few weeks ago my friend Sasha Karabegov acquinted me with the problems suggested on the PUTNAM competition in USA universities [1)]. Sasha was local organiser of this competition in his University. He has suggested the beautiful solutions of some questions, and in particular of the question A5 on this competition [2)]. His elegant proof is based on the identity (2). In fact in this proof he deduces in elementary way the following Theorem:

Let $f(x)$ be a smooth function on $\mathbf{R}$ such that this function and its all derivatives at all the points take non-negative values. Then the condition that function f vanishes at arbitrary point implies that it vanishes at all the points:

$$\forall x, \forall n f^{(n)}(x) \geq 0 \,\text{and}\; f(x_0) = 0 \Rightarrow f(x) \equiv 0\,. \tag{3}$$

This statement is related with the Bernstein's Theorem on monotone functions[3)].

In what follows I expalin the identity (1) and reproduce Karabegov's proof.

Taylor identity

I have known this identity ' for hundred years'. Karabegov's proof makes me to realise that this is really very effective.

Let $f = f(x)$ be a smooth function. Then integrating by parts we come to

$$f(x) = f(0) + \int_0^x \frac{df(t))}{dt} dt = f(0) + \underbrace{\int_0^x \frac{df(t))}{dt} \cdot 1 dt}_{\mathrm{I}} =$$

1) see https://kskedlaya.org/putnam-archive/2018.pdf
2) see the fourth solution of this question in https://kskedlaya.org/putnam-archive/2018s.pdf or the Appendix to this text
3) see the second solution of the question in https://kskedlaya.org/putnam-archive/2018s.pdf

$$f(0)+\int_0^x \frac{d}{dt}f(t)\frac{d}{dt}\,(t-x)\,dt = f(0)+\frac{d}{dt}f(t)(t-x)\big|_0^x-\int_0^x \frac{d^2}{dt^2}f(t)\,(t-x)\,dt =$$

$$f(0)+f'(0)x+\underbrace{\int_0^x \frac{d^2}{dt^2}f(t)\,(x-t)\,dt}_{\text{II}} =$$

$$f(0)+f'(0)x-\frac{1}{2}\int_0^x \frac{d^2}{dt^2}f(t)\frac{d}{dt}\left((t-x)^2\right)dt =$$

$$f(0)+f'(0)x-\frac{1}{2}\frac{d^2}{dt^2}f(t)\,(t-x)^2\,\big|_0^x+\frac{1}{2}\int_0^x \frac{d^3}{dt^3}f(t)\,(t-x)^2\,dt =$$

$$f(0)+f'(0)x+f^{''}(0)\frac{x^2}{2}+\underbrace{\frac{1}{2}\int_0^x \frac{d^3}{dt^3}f(t)\,(x-t)^2\,dt}_{\text{III}} =$$

$$f(0)+f'(0)x+f^{''}(0)\frac{x^2}{2}-\frac{1}{6}\int_0^x \frac{d^3}{dt^3}f(t)\frac{d}{dt}\left((x-t)^3\right)dt =$$

$$f(0)+f'(0)x+f^{''}(0)\frac{x^2}{2}+\frac{1}{6}\frac{d^3}{dt^3}f(t)\,(t-x)^3\,\big|_0^x-\frac{1}{6}\int_0^x \frac{d^4}{dt^4}f(t)\,(x-t)^3\,dt =$$

$$f(0)+f'(0)x+f^{''}(0)\frac{x^2}{2}+f^{'''}(0)\frac{x^3}{6}+\underbrace{\frac{1}{6}\int_0^x \frac{d^4}{dt^4}f(t)\,(x-t)^3\,dt}_{\text{IV}} =$$

$$f(0)+f'(0)x+f^{''}(0)\frac{x^2}{2}+f^{'''}(0)\frac{x^3}{6}-\frac{1}{24}\int_0^x \frac{d^4}{dt^4}f(t)\frac{d}{dt}\left((t-x)^4\right)dt =$$

$$f(0)+f'(0)x+f^{''}(0)\frac{x^2}{2}+f^{'''}(0)\frac{x^3}{6}-\frac{1}{24}\frac{d^4}{dt^4}f(t)\,(t-x)^4\,\big|_0^x+\frac{1}{24}\int_0^x \frac{d^5}{dt^5}f(t)\,(t-x)^4\,dt =$$

$$f(0)+f'(0)x+f^{''}(0)\frac{x^2}{2}+f^{'''}(0)\frac{x^3}{6}+f^{''''}(0)\frac{x^4}{24}+\underbrace{\frac{1}{24}\int_0^x \frac{d^5}{dt^5}f(t)\,(x-t)^4\,dt}_{\text{IV}} =$$

and so on:

$$\ldots=\sum_{k=1}^{n} f^{(k)}(0)\frac{x^k}{k!}+\frac{1}{n!}\int_0^x \frac{d^{n+1}}{dt^{n+1}}f(t)(x-t)^n dt$$

Appendix

`Here I reproduce the Karabegov's proof of the Theorem (3)..`

Shortly speaking his proof is the following: if $f(x_0) = 0$ then for all $x \le x_0$, $f(x) = 0$ also since $f'(x) \ge 0$ for $x \le x_0$. Thus all derivatives of the smooth function f vanish at the point x_0. Hence it follows from the identity (1) that for all x and for all n,

$$f(x) = \frac{1}{n!} \int_{x_0}^{x} \frac{d^{n+1}}{dt^{n+1}} f(t)(x-t)^n dt \,. \tag{A1}$$

for an arbitary n. Hence integrating this identity we see that for every x_1 and for every n

$$\int_{x_0}^{x_1} f(x)dx = \int_{x_0}^{x_1} dx \left(\frac{1}{n!} \int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n dt \right) =$$

$$\frac{1}{n!} \int_{x_0}^{x_1} dt \left(\int_{t}^{x} f^{(n+1)}(t)(x-t)^n dx \right) = \frac{1}{(n+1)!} \int_{x_0}^{x_1} f^{(n+1)}(t)(x_1-t)^{n+1} \,. \tag{A2}$$

Choose an arbitrary $x_1 > x_0$. Then it follows from equations (A2) and (A1) that

$$\int_{x_0}^{x_1} f(x)dx = \frac{1}{(n+1)!} \int_{x_0}^{x_1} f^{(n+1)}(t)(x_1-t)^{n+1} \le$$

$$\frac{x_1-x_0}{n+1} \left(\frac{1}{n!} \int_{x_0}^{x_1} f^{(n+1)}(t)(x_1-t)^n \right) = \frac{x_1-x_0}{n+1} f(x_1) \Rightarrow f(x_1) = 0 \quad \blacksquare$$

since this inequality holds or arbitrary n.