**Three identities on determinants**

**1.** Let $B$ be $m\times n$ matrix and $D$ be $n\times m$ matrix. Then

$$\det(1+BD)=\det(1+DB)\,. \tag{1}$$

*Proof.* $\mathrm{Tr}\,(BD)^k=\mathrm{Tr}\,(DB)^k$. Hence characteristic polynomials $\det(1+zBD)$ and $\det(1+zBD)$ coincide. Thus we come to (1). How to prove it in another way?

It follows from (1) that if $D=B^+=(x^1,x^2,\dots,x^n)$ is $1\times n$ matrix then

$$\det(\delta^{ik}+x^ix^k)=\det(1+BD)=\det(1+DB)=\det(1+x^ix^i)=1+(x^1)^2=\ldots+(x^n)^2$$

The relation (1) has two very amusing "generalisation":

**2.** Assume that entries of matrices $B$ and $D$ *are "odd numbers", i.e. anticommuting elements of some* $\mathbf{Z_2}$*-algebra (e.g. odd elements of Grassmann algebra).* In this case we have instead (1) the following elation:

$$\det(1+\mathcal{B}\mathcal{D})=\frac{1}{\det(1+\mathcal{D}\mathcal{B})} \tag{2}$$

(Here and later denoting $\mathcal{B}$ instead $B$ and $\mathcal{D}$ instead $D$ I would like to emphasize the odd nature of matrices.)

To prove (2) we note that $\mathrm{Tr}\,(\mathcal{B}\mathcal{D})^k=-\mathrm{Tr}\,(\mathcal{D}\mathcal{B})^k$. Hence

$$\frac{d}{dz}\log\left[\det(1+zBD)\det(1+zDB)\right]=\mathrm{Tr}\,\left[(1+zBD)^{-1}BD+(1+zDB)^{-1}DB\right]=$$

$$\sum_k z^k\left[\mathrm{Tr}\,(BD)^{k+1}+\mathrm{Tr}\,(DB)^{k+1}\right]=0\ \Rightarrow \det(1+zBD)\det(1+zDB)\equiv 1\,.$$

**3.**. If $\mathcal{B}$ and $\mathcal{D}$ are $n\times n$ matrices such that $\mathcal{B}$ is symmetrical matrix and $\mathcal{D}$ is antisymmetrical matrix ($\mathcal{B}_{ik}=\mathcal{B}_{ki}$, $\mathcal{D}_{ik}=-\mathcal{D}_{ki}$) then

$$\det(1+\mathcal{B}\mathcal{D})=1\ , \tag{3}$$

This is very important identity*.

This identity can be considered as a special case of identity (2). But it can be proved independently. Indeed to prove the identity (2) it is enough to show that

$$\mathrm{Tr}\ (\mathcal{B}\mathcal{D})^k=0\,, \tag{4}$$

since in the same way as above the relation (4) implies that characteristic polynomial $\det(1+z\mathcal{B}\mathcal{D})$ equals to 1. We have:

$$\mathrm{Tr}\,(\mathcal{B}\mathcal{D})^k=-\mathrm{Tr}\,(\mathcal{D}\mathcal{B})^k=\mathrm{Tr}\,(\mathcal{D}^+\mathcal{B}^+)^k=-\mathrm{Tr}\ \left((\mathcal{B}\mathcal{D})^+\right)^k=-\mathrm{Tr}\,(\mathcal{B}\mathcal{D})^k.$$

Hence we come to (4) ∎

* In particular it follows form this identity that square root of Berezinian (superdeterminant) of linear canonical transformation is equal to the determinant of its boson-boson sector