$$\tilde{\mathbf{Z}}_{10} = \mathbf{Z}_2 \oplus \mathbf{Z}_5$$

It is very natural to consider 10-adic numbers in spite of the fact that 10 is not prime.

The simplest way to define the ring $\tilde{\mathbf{Z}}_{10}$ is the following: The set $\tilde{\mathbf{Z}}_{10}$ is the set of formal series $\sum_{n=0}^{\infty} a_n 10^n$ where a_n are numbers $\{0,1,2,3,4,5,6,7,8,9\}$. One can naturally introduce ring structure mimicking addition and multiplication rules, so called "slozhenije i umnozhenije stolbikom". E.g. If $x = \sum_{n=0}^{\infty} a_n 10^n$, $y = \sum_{n=0}^{\infty} b_n 10^n$ then $x+y = \sum_{n=0}^{\infty} c_n 10^n$, where c_n are defined by

$$c_n = \begin{cases} a_n + b_n + r_n \text{ if } a_n + b_n + r_n < 10 \\ a_n + b_n + r_n - 10 \text{ if } a_n + b_n + r_n \geq 10 \end{cases} \quad n = 0, 1, 2, \ldots$$

and

$$r_0 = 0,\ r_{k+1} = \begin{cases} 0 \text{ if } a_n + b_n + r_k < 10 \\ 1 \text{ if } a_n + b_n + r_n \geq 10 \end{cases} \text{ for } k \geq 0$$

To be more precise consider presentation of p-adic numbers by infinite sequence: $x = (x_0, x_1, x_2, \ldots, x_n)$ where $x_0 = a_0, x_1 = a_0 + 10a_1, \ldots, x_k = \sum_{n=0}^{k} a_n 10^n$, ... if $x = \sum_{n=0}^{\infty} a_n 10^n$. The natural projection $P_k \colon \sum_{n=0}^{\infty} a_n 10^n \to \sum_{n=0}^{k} a_n 10^n$ projects x_{k+1} on x_k. One can see that if $x = \sum_{n=0}^{\infty} a_n 10^n = (x_0, x_1, x_2, x_3, \ldots)$, $y = \sum_{n=0}^{\infty} b_n 10^n = (y_0, y_1, y_2, y_3, \ldots)$ then

$$x + y = z = (z_0, z_1, z_2, z_3, \ldots)$$

where $z_k = P_k(x_k + y_k)$ and

$$xy = w = (w_0, w_1, w_2, w_3, \ldots)$$

where $w_k = P_k(x_k y_k)$ This ring is not integral domain: (see examples below).

It is well=known that If integer N is product of different primes then $\mathbf{Z}/N\mathbf{Z}$ is direct sum of fields. In particular $\mathbf{Z}/10\mathbf{Z} = \mathbf{F_2} + \mathbf{F_5}$, (here as always $\mathbf{F_p} = \mathbf{Z/pz}$ prime field of characteristic p for prime p)— The China‘s algorithm establish the isomorphism between $\mathbf{F_{p_1}} \oplus \mathbf{F_{p_2}}$ and $\mathbf{Z}/p_1p_2\mathbf{Z}$ if $p_1 \neq p_2$.

This can be be prolonged:

Proposition *The ring $\tilde{\mathbf{Z}}_{10}$ is isomorphic to the direct sum of the rings $\mathbf{Z}_2$ and $\mathbf{Z}_5$.*

Present explicitly the maps $\phi \colon \mathbf{Z}_{10} \to \mathbf{Z}_2 \oplus \mathbf{Z}_5$ and inverse map $\psi \colon \mathbf{Z}_2 \oplus \mathbf{Z}_5 \to \mathbf{Z}_{10}$ which establish this isomorphism. If $x = \sum_{n=0}^{\infty} a_n 10^n \in \mathbf{Z}_{10}$ then

$$\psi(x) = \left(\sum_{n=0}^{\infty} (5^n a_n)\, 2^n, \sum_{n=0}^{\infty} (2^n a_n)\, 5^n \right) \in \mathbf{Z}_2 \oplus \mathbf{Z}_5$$

Note that this map sends rational integrals on the diagonal: Image of ϕ on $\mathbf{Z}$ is diagonal in $\mathbf{Z} \oplus \mathbf{Z}$.

The inverse map is little bit not so obvious:

Let $(x, y) \in \mathbf{Z}_2 \oplus \mathbf{Z}_5$ where $x = (x_0, x_1, x_2, \ldots,)$ with $x_k = \sum_{n=0}^{k} a_n 2^n$ and $y = (y_0, y_1, y_2, \ldots,)$ with $y_k = \sum_{n=0}^{k} b_n 5^n$. Show that there exists $z = (x_0, x_1, x_2, \ldots,)$ with $z_k = \sum_{n=0}^{k} c_n 10^n$ which obeys the conditions:

$$z_k = x_k (\mathrm{mod}\, 2^{k+1}), \quad z_k = y_k (\mathrm{mod}\, 5^{k+1}), k = 0, 1, 2, 3, \ldots$$

and z_k are uniquely defined by these conditions. It follows from this statement that map $\psi = \phi^{-1}$ is defined by equation $\psi(x, y) = z$

For $k = 0$ this is obvious. Suppose we proved it for $k \leq l$. For $k = l + 1$ we have equations

$$z_{l+1} = z_l + c_{l+1} 10^{l+1} = x_{l+1} = x_l + a_{l+1} 2^{l+1} (\mathrm{mod}\, 2^{l+2})$$

and

$$z_{l+1} = z_l + c_{l+1} 10^{l+1} = y_{l+1} = y_l + b_{l+1} 5^{l+1} (\mathrm{mod}\, 5^{l+2})$$

on unknowns $a_{l+1}, b_{l+1} \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. These equations have solutions and these solution is unique because due to inductive hypothesis for $k = l$ $z_l = x_l + \delta^{(2)} 2^{l+1}$ and $z_l = x_l + \delta^{(5)} 5^{l+1}$.

Remark The maps ϕ, ψ are "lifting" of the maps establishing isomorphism between ring $\mathbf{Z}/10\mathbf{Z}$ and direct sum of the fields $\mathbf{F_2}$ and $\mathbf{F_5}$.

Example. Consider an elements $(1, 0) \in \mathbf{Z}_2 \oplus \mathbf{Z}_5$. Find an element $z = \sum_{n=0}^{\infty} c_n 10^n$ such that $\phi(z) = (1, 0)$. If $z = (z_0, z_1, z_2, z_3, \ldots)$ then one can see that

$$z_0 = 5, z_1 = 25, z_2 = 625, \ldots$$

we come to.... Yes! you are right we come to the sequence which we know very well from the school: sequence $5, 25, 625, \ldots$—sequence of numbers such that final digits of their squares coicide with these numbers: $5^2 = \mathbf{25}$, $25^2 = 6\mathbf{25}$,.. In the language of 10-adic numbers $z = \psi(1.0)$ is 10-adic number $(5, 25, 625, \ldots)$ such that $x^2 = x$. It is because

$$x^2 = (1, 0)^2 = (1, 0) = x$$

Now we see that the second non-trivial solution of the equation $x^2 = x$ in the ring $\mathbf{Z}_2 \oplus \mathbf{Z}_5$ is $w = (0, 1)$. One can see that

$$\psi(0, 1) = (6, 76, 376, \ldots)$$

Remark The following simple and beautiful interpretation : equation $x^2 = x$ has more than one solution since 10 is not a prime number*: $\mathbf{Z}_{10}$ is not an integral domain; e.g. $(a, 0)(0, b) = (0, 0) = 0$. $(1, 0)(0, 1) = 0$

Pseudo Teichmullers for $\mathbf{Z}_{10}$

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* This interpretation belongs to A.Veselov.

Recall standard facts about Teichmuller map. If p is prime then the ring $\mathbf{Z}_p$ possesses all roots of degree $p-1$ of unity, i.e. there exist a map $T\colon \mathbf{F_p} \to \mathbf{Z_p}$ (Teihmuller map) such that

$$T^p(\bar{a}) = T(a),$$

One can see that $T(\bar{a}\bar{b}) = T(\bar{a})T(\bar{b})$ and $T(\mp\bar{a}\bar{b}) = T(\bar{a}) + T(\bar{b}) = p\mathbf{Z}_p$ Here $\mathbf{F_p} = \{\bar{\mathbf{0}}, \bar{\mathbf{1}}, \bar{\mathbf{2}}, \dots\}$ is a prime field of characteristic p.

Roughly speaking for any $a \in 1, 2, \dots, p-1$ there is p-adic number, i.e. a sequence $\{x_0, x_1, x_2\}$ such that $x_{k+1} = x_k (mod p^k)$ and $x_k^p = x_k + \dots$. One can see that

$$T(\bar{a}) = \lim_{n\to\infty} a^{p^n} = (a, a^p, a^{p^2}, \dots, a^{p^n}, \dots)$$

because $a^{p^{n+1}} = a^{p^{n+1}-p^n} a^{p^n}$ and $a^{p^{n+1}-p^n} = a^{\varphi(p^n)} = 1 + ...p^{n+1}$

What happens in $\tilde{\mathbf{Z}}_{10}$? We already know that for $\bar{a} = 5, 6 \in \mathbf{Z}/10\mathbf{Z}$ $\tilde{T}(\bar{5}) = (5, 25, \dots)$ and $\tilde{T}(6) = (6, 76, \dots)$ the order of these elements is equal to 2.

Now look for all elements

$\bar{a} = 1$ $\tilde{T}(\bar{1}) = 1$. Order is equal to 2 $(1^2 = 1)$

$\bar{a} = \bar{2}$. We have that $\bar{2} = (\bar{0}, \bar{2})$ in $\mathbf{F_2} \oplus \mathbf{F_5}$. We see that $\tilde{T}(\bar{2}) = (0, T_5(\bar{2}) = \big(0, 2^{5^\infty}\big)$. $2^{5^\infty} = (2, 32,)$ Its image in $\tilde{\mathbf{Z}}_{10}$ is equal to 2^{5^∞}

In the pedestrians language we come to the sequence:

$(2, 32, 432, \dots)$ such that $2^5 = \mathbf{32}$, $32^5 = ...4\mathbf{32}$, $432^5 = ...4\mathbf{432}$

$\bar{a} = \bar{3}$. We have that $\bar{3} = (\bar{1}, \bar{3})$ in $\mathbf{F_2} \oplus \mathbf{F_5}$. We see that $\tilde{T}(\bar{3}) = (1, T_5(\bar{2}) = \big(0, 3^{5^\infty}\big)$. $(T_2(1) = 1)$. $3^{5^\infty} = (3, \dots 43, \dots 443, \dots)$, $3^5 = ...43$, $43^5 = ...443$

Now we have to calculate the image of $(1, 3^{5^\infty})$ in $\tilde{\mathbf{Z}}_{10}$. Is it equal to 3^{5^∞}? Yes

In the pedestrians languange we come to the sequence:

$(3, 43, 443, \dots)$ such that $3^5 = 4\mathbf{3}$, $43^5 = ...4\mathbf{43}$, $443^5 = ...\mathbf{443}$

$\bar{a} = \bar{4}$. We have that $\bar{4} = (\bar{0}, \bar{4})$ in $\mathbf{F_2} \oplus \mathbf{F_5}$. Note that order of $\bar{4}$ in $\mathbf{F_5}$ is equal to 2. We see that $\tilde{T}(\bar{4}) = (0, T_5(\bar{4}) = \big(0, 4^{5^\infty}\big)$. $4^{5^\infty} = (4, \dots 24, \dots 624, \dots)$, $4^5 = ...2\mathbf{4}$, $24^5 = ...6\mathbf{24}$, $624^5 = \dots\mathbf{624}$... In fact cubes not only fifth orders have the same end: $4^3 = ...\mathbf{4}$, $24^5 = ...\mathbf{24}$, $624^5 = \dots\mathbf{624}$...

The image of $(0, 4^{5^\infty})$ in $\tilde{\mathbf{Z}}_{10}$. Is it equal to 4^{5^∞}? Yes

In the pedestrians languange we come to the sequence:

$(4, 24, 624, \dots)$ such that $4^3 = ...4$, $24^3 = ...24$,... and

$4^5 = 4\mathbf{24}$, $24^5 = ...6\mathbf{243}$,

$\bar{a} = \bar{5}$. We have that $\bar{5} = (\bar{1}, \bar{0})$ in $\mathbf{F_2} \oplus \mathbf{F_5}$ We know already that $\tilde{T}(5) = (5, 25, 625, \dots)$. 10-adic number $x = 5^{2^\infty} = (5, 25, 625, \dots)$ obeys the equation $x^2 = x$

$\bar{a} = \bar{6}$. We have that $\bar{6} = (\bar{0}, \bar{1})$ in $\mathbf{F_2} \oplus \mathbf{F_5}$ We know already that $\tilde{T}(6) = (6, 76, 376, \dots)$. 10-adic number $y = 6^{5^\infty} = (6, 76, 376, \dots)$ obeys the equation $x^2 = x$

$\bar{a} = \bar{7}$. We have that $\bar{6} = (\bar{1}, \bar{2})$ in $\mathbf{F_2} \oplus \mathbf{F_5}$ 10-adic number $x = 7^{5^\infty}$ obeys the equation $x^5 = x$.

$\bar{a} = \bar{8}$. We have that $\bar{6} = (\bar{0}, \bar{3})$ in $\mathbf{F_2} \oplus \mathbf{F_5}$ 10-adic number $x = 8^{5^\infty}$ obeys the equation $x^5 = x$.

$\bar{a} = \bar{9}$. We have that $\bar{9} = (\bar{1}, \bar{4})$ in $\mathbf{F_2} \oplus \mathbf{F_5}$ 10-adic number $x = 9^{5^\infty}$ obeys the equation $x^3 = x$ (and $x^5 = x$).