14 November 2018.

The file below was written more that six years ago. It is about the elegant (at least it seems to me such) way to calculate the number

$$S_k = \#\{\texttt{number of permutations of } k \texttt{ elements which replace every element}\}.$$

E.g. $S_1 = 0, S_2 = 1$, $S_3 = 2$.

Few days ago I realise that the Z_2-verion of the file (that is information about that S_2 is odd or even) has a meaning for calculating determinants (see the etude linearalgebr.tex). I present it here as the addendum to the main text.

One combinatorial problem

Autumn, 2012

It was long long long ago when I solved the following exercise: Denote by S_k a number of sequences of n natural numbers $\{1, 2, 3, \ldots, k\}$ such that all the numbers are on the wrong places, i.e. the first number is not 1, the second number is not 2, e.t.c.

I forget the calculations. I just rememeber that they were not nice, but the answer was beautiful, something like $S_k \approx k!/e$. Two months ago I found a following beautiful solution. Here it is:

$$\text{One can see that} \quad \sum_{k=1}^{n} C_n^k S_k = n!\,, \tag{1}$$

where $C_n^k = \begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{k!(n-k)!}$. (The right hand sight of equation (1) is the number of all permutations of the set with n elements. The summand $C_n^k S_k$ in the left hand side is the number of permutations such that exactly $n - k$ elements are fixed*.)

Recall that the n-th derivative of the product FG of two functions F and G is given by the formula

$$\left(\frac{d^n}{dx^n}\right)(F(x)G(x)) = \sum_{k=1}^{n} C_n^k \left(\frac{d}{dx}\right)^k (F(x)) \left(\frac{d}{dx}\right)^{n-k} (G(x))\ .$$

Comparing this formula with relation (1) we see that if we choose

$$F = \sum \frac{S_k}{k!} x^k\,, \qquad \text{and } G = e^x\,,$$

then

$$\left(\frac{d^n}{dx^n}\right)(F(x)G(x))_{x=0} = \left(\frac{d^n}{dx^n}\right)(F(x)e^x)_{x=0} = \sum_{k=1}^{n} C_n^k S_k = n!\,.$$

* One can see that it is reasonable to assume that $S_0 = 1$

Hence

$$F(x)e^x = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

in a vicinity of $x = 0$. We come to the answer: the sequence S_k is such that

$$F = \sum_{k=0}^{\infty} \frac{S_k}{k!} = \frac{e^{-x}}{1-x}\,.$$

Using this formula we write down the explicit formula for S_k. Denote by $s_k = \frac{S_k}{k!}$. We have that

$$\sum_{k=0}^{\infty} \frac{S_k}{k!} x^k = \sum_{k=0}^{\infty} s_k x^k = \frac{e^{-x}}{1-x} = \left(1 - x + \frac{x^2}{2} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \dots\right)(1 + x + x^2 + x^3 + \dots)$$

$$= 1 + (1-1)\,x + \left(1 - 1 + \frac{1}{2!}\right)x^2 + \left(1 - 1 + \frac{1}{2!} - \frac{1}{3!}\right)x^3 + \left(1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right)x^4 + \dots$$

i.e.

$$s_k = \frac{S_k}{k!} = \sum_{p=0}^{k} \frac{(-1)^p}{p!}, \text{ and } S_k = k! \sum_{p=0}^{k} \frac{(-1)^p}{p!}\,. \tag{2}$$

In particular

$$s_\infty = \lim_{k\to\infty} = \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} = \frac{1}{e}\,, \tag{3}$$

i.e. the probability that all terms of the sequence $\{1, 2, 3, \dots, N\}$ are on the wrong places equals to $\approx \frac{1}{e}$ when $N \to \infty$.

Is S_k even or odd?

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This question is in particular important e.g. in linear algebra

We can straightforwardly answer this question, analyzing the formula (2):

$$S_k \text{ is even if } k \text{ is odd, and } S_k \text{ is odd if } k \text{ is even} \tag{4}$$

This is much better to come to the answer just analyzing the formula (1) over the field Z_2. Notice that

$$\sum_{\text{even } k} C_n^k = \sum_{\text{odd } k} C_n^k = 2^{n-1}\,. \tag{5}$$

Now it has just to check that the sequence (4) obeys the equations

It is evident that for every N, the sysem of $N+1$ equations

$$\sum_{i=0}^{n} C_n^i S_i = n!\,,\text{i.e.} \begin{cases} C_0^0 S_0 = S_0 = 1 \\ C_1^0 S_0 + C_1^1 S_1 = S_0 + S_1 = 1 \\ C_2^0 S_0 + C_2^1 S_1 + C_2^2 S_2 = S_0 + 3S_1 + S_2 = 2 \\ C_3^0 S_0 + C_3^1 S_1 + C_3^2 S_2 + C_3^3 S_3 = S_0 + 3S_1 + 3S_2 + S_3 = 6 \\ \dots \end{cases} \tag{5}$$

on $N+1$ variables $S_0, S_1, S_2, \dots S_N$ has unique solution $S_0 = 1$, $S_1 = 0$, $S_2 = 1$,$S_3 = 3$,.... Hence it has unique solution in the field Z_2. On the other hand the sequence identity (5) implies that the sequence (4) is the solution of equations (5). This proves that all S_k have parity $k+1$.