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Sum $1+2+3+\ldots=-\frac{1}{12}$

Yesterday David showed me the file in 'Youtube' where it is given very elegant 'proof' of this relation (see http://goo.gl/bYh4DL) I decided to write down here the standard calculations which lead to this result. Here they are

We present here calculation of analytical continuation of Riemann ζ-function. In particular we come to Euler formula of calculation of ζ-function at negative integers. The title of this étude = value of ζ-function at $s=-1$.

Recall that for Γ-function

$$\Gamma(s)=\int_0^\infty t^{s-1}e^{-t}dt\,. \tag{1a}$$

It obeys the relation

$$\Gamma(s+1)=s\Gamma(s)\,. \tag{1b}$$

These relations define Γ-function for all complex plane. Since $\Gamma(1)=1$ hence due to (1b) $\Gamma(n+1)=n!$. One can see that that Γ-function has poles at all non-positive integers. Indeed

$$\Gamma(s)=\frac{s(s+1)(s+2)...(s+n)\Gamma(s)}{s(s+1)...(s+n)}=\frac{\Gamma(s+n+1)}{s(s+1)\ldots(s+n)}\,.$$

Hence in a vicinity of an arbitrary point s, for $s=-n+\varepsilon$ $(n=0,-1,-2,\ldots)$

$$\Gamma(s)=\Gamma(-n+\varepsilon)=\frac{\Gamma(1+\varepsilon)}{(-n+\varepsilon)(-n+1+\varepsilon)\ldots(-1+\varepsilon)\varepsilon}=\frac{(-1)^n}{n!}\left(\frac{1}{\varepsilon}+\ldots\right)\,. \tag{1c}$$

Express ζ-function, $\zeta(s)=\sum_{n=1}^\infty\frac{1}{n^s}$ in terms of Γ-function.

It follows from (1a) that

$$\zeta(s)=\sum_{n=1}^\infty\frac{1}{n^s}=\sum_{n=1}^\infty\left(\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-nt}dt\right)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\frac{e^{-t}dt}{1-e^{-t}} \tag{2}$$

Consider an expansion of function $\frac{e^{-t}}{1-e^{-t}}$ in a vicinity of point $t=0$:

$$\frac{e^{-t}}{1-e^{-t}}=\sum_{k=-1}^\infty\Psi_k t^k=\frac{1}{t}-\frac{1}{2}+\frac{t}{12}-\frac{t^3}{720}+\ldots\,. \tag{3}$$

Remark Notice that all coefficients Ψ_{2k} in this expansion vanish for $k\geq 1$ since

$$\frac{e^{-t}}{1-e^{-t}}=-\frac{1}{2}+\underbrace{\frac{1}{2}\text{cotan}\,\frac{t}{2}}_{\text{even function}}\,. \tag{3b}$$

Now using expansion (3) we see that right hand side in (2) is convergent for $\Re s > 2$:

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{e^{-t}dt}{1-e^{-t}} = \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} \frac{e^{-t}dt}{1-e^{-t}} + \frac{1}{\Gamma(s)} \int_1^\infty t^{s-1} \frac{e^{-t}dt}{1-e^{-t}} =$$

$$= \frac{1}{\Gamma(s)} \sum_{k\geq -1} \int_0^1 \Psi_k t^{s+k-1} dt + \frac{1}{\Gamma(s)} \int_1^\infty t^{s-1} \frac{e^{-t}dt}{1-e^{-t}} .$$

For $\Re s > 2^*$

$$\zeta(s) = \sum \frac{1}{n^s} = \sum_{k=-1}^\infty \frac{\Psi_k}{(k+s)\Gamma(s)} + \frac{1}{\Gamma(s)} \int_1^\infty t^{s-1} \frac{e^{-t}dt}{1-e^{-t}} . \tag{4}$$

Now note that the integral in this expression is analytical function for all s. Hence equation (4) defines analytical continuation, meromorphic ζ-function for all s.

We focus at the values of $\zeta(s)$ at non-positive integers. Non-positive integers are poles of Γ-function (see 1(c)). Hence the second integral vanishes at these points. Due to relation (1c) we have that all terms which are proportional to $\frac{1}{\Gamma(s)(k+s)}$ for $k \neq m$ vanish too at the point $s = -m$. We have from (1c) that

$$\zeta(-m) = \sum_{k=1}^{-\infty} \frac{\Psi_k}{\Gamma(s)(s+k)}\Big|_{s=-m} = \frac{\Psi_m}{\Gamma(s)(s+m)}\Big|_{s=-m} .$$

Now recall the behaviour of Γ-function in a vicinity of the pole $s = -m$. Due to (1c) $\Gamma(s) = \Gamma(-m+\varepsilon) = \frac{(-1)^m}{m!}\left(\frac{1}{\varepsilon} + O(1)\right)$ and

$$\zeta(-m) = \frac{\Psi_m}{\Gamma(s)(s+m)}\Big|_{s=-m} = \lim_{\varepsilon\to 0} \frac{\Psi_m}{\Gamma(-m+\varepsilon)\varepsilon} = (-1)^m m! \Psi_m . \tag{5}$$

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* To calculate analytical continuation we represented the integral $\int_0^\infty$ as a sum of integrals $\int_0^1$ and $\int_1^\infty$. One can show that the function $\frac{e^{-t}}{1-e^{-t}}$ can be replaced under integral $\int_0^1$ by series $\sum \Psi_k t^k$ in spite of the fact that there is no convergence at the point $t = 1$. To avoid this additional work one may take an arbitrary a: $0 < a < 1$ and represent the integral as a sum of integrals $\int_0^a$ and $\int_a^\infty$. In this case we have no any problem with series convergence: $\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{e^{-t}dt}{1-e^{-t}} =$

$$= \frac{1}{\Gamma(s)} \int_0^a t^{s-1} \frac{e^{-t}dt}{1-e^{-t}} + \frac{1}{\Gamma(s)} \int_a^\infty t^{s-1} \frac{e^{-t}dt}{1-e^{-t}} = \sum_{k=-1}^\infty \frac{\Psi_k a^{k+s}}{(k+s)\Gamma(s)} + \frac{1}{\Gamma(s)} \int_a^\infty \frac{t^{s-1}e^{-t}dt}{1-e^{-t}}$$

One can see that we again come to (5) since only the term which possesses $a^0 = 1$ gives contribution to the integral. (Intersting remark for curious reader that the answer does not depend on a choice of a).

This is the answer. In particular due to (3b)

$$\zeta(-2m) = 0\,, \quad \text{for } m \geq 1 \tag{5b}$$

We have

$$\zeta(0) = \Psi_0 = -\frac{1}{2}\,, \quad \zeta(-1) = -\Psi_1 = -\frac{1}{12}\,, \quad \zeta(-2) = 0\,, \quad \zeta(-3) = -6\Psi(3) = \frac{1}{120}\,, \zeta(-4) = 0,$$

and so on,

One can say that in Pickwickian sense

$$1^m + 2^m + 3^m + \ldots + k^m + \ldots \text{`}=' \; m!(-1)^m \Psi_{-m}\,,$$

in particular we come to very famous in string theory paradox:

$$1 + 2 + 3 + \ldots + k + \ldots \text{`}=' -\Psi_{-1} = -\frac{1}{12}\,.$$

Note that relation (5b) describes all so called *trivial zeros of* ζ function.