Galois theory for pedestrians

Talk on Galois group on Wednesday 2-nd March at 1 pm

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Short abstract:

We demonstrate some basic ideas of Galois Theory considering quadratic cubic and quartic equations.

Detailed abstract

It is well-known that roots x_1, x_2 of quadratic polynomial $P(x) = x^2 + px + q$ obey the condition

$$\begin{cases} x_1 + x_2 = -p \\ x_1 x_2 = q \end{cases}, \quad \text{(Vieta's formula)}\,.$$

Conisder another functions on roots. E.g. one can see that

$$x_1^4 + x_2^4 = p^4 - 4p^2 q + 2q^2\,,$$

but

$$x_1^4 - x_2^4 = \pm(2pq - p^3\sqrt{p^2 - 4q}\,.$$

The reason why the first expression is polynomial of roots, and second expression is proportional to a square root, is the following: the first expression is invariant with respect to the permutation $x_1 \leftrightarrow x_2$, or more formally with respect to the action of group S_2 of permutation on roots, and the second expression changes the sign under action of permutation $x_1 \to x_2$.

This statement can be generalised. Let $x_1, \ldots, x_n$ be roots of n-th order polynomial

$$P(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n\,, \tag{1}$$

where $a_1, \ldots, a_n$ are parameters. One can consider the group of permutations of the roots which preserve the coefficients of the polynomnial—Galois group. In the case of polynomial (1) it is just a group S_n of permutations of roots.

It turns out that properties of roots of polynomial such as

expressability in radicals (problem of solution)

Does given expression is a polynomial on parameters $a_1, \ldots, a_n$ or not?

Does given expression is expressed via square root operation?

and many others

can be expressed in answered in terms of the Galois group of the polynomial.

For example if x_1, x_2, x_3 are roots of cubic polynomial

$$x^3 + px + q = 0\,,$$

then the expression

$$A = A(x_1, x_2, x_3) = (x_1^5 + x_2^5)(x_2^5 + x_3^5)(x_3^5 + x_1^5)\,,$$

is a polynomial on parameters p, q since this expression is invariant under all permutations of roots, and the expression

$$B = B(x_1, x_2, x_3)i = (x_1^5 - x_2^5)(x_2^5 - x_3^5)(x_3^5 - x_1^5)$$

is expressed via coefficients p, q and square root operation, since this expression takes two values under the action of group S_3 of all permutations of roots.

We answer these questions studying the action of group S_3 of permutations of roots on RHS of these expressions.

In this talk we try to explain these ideas and on the base of these considerations we will come to the formulae for solutions of cubic and quartic equations