Lagrange Theorem

Let $K_n = K[x_1, x_2, \ldots, x_n]$ be a ring of polynomials over field K where $x_1, \ldots, x_n$ are free variables (independent indeterminates) and $K(x_1, x_2, \ldots, x_n)$ be a corresponding field of fractions.

Let $G_n[x_1, \ldots, x_n]$ be a subring of symmetrical polynomials. Consider basic symmetric polynomials

$$t_k = x_1^k + x_2^k + \ldots + x_n^k\,, \quad k = 1, 2, 3, \ldots, n\,. \tag{1}$$

All elements of subring $G_n[x_1, \ldots, x_n]$ are polynomials on $t_1, \ldots, t_n$.

Every polynomial F has its invariance subgroup $H_F \leq S_n$ of the symmetric group S_n: permutation $\sigma \in S_n$ belongs to H_F iff polynomial F remains invariant under this permutation: $F^\sigma = F$.

Theorem (Lagrange): Let $F = F(x_1, \ldots, x_n), G = G(x_1, \ldots, x_n)$ be two polynomials in $K[x_1, \ldots, x_n]$ such that H_F is a subgroup of H_G. Then polynomial G can be considered as rational function on symmetric polynomials $t_1, \ldots, t_n$ and polynomial F. More formally there exists a fraction $\frac{P(z,t_1,\ldots,t_n)}{Q(z,t_1,\ldots,t_n)}$ such that

$$G(x_1, \ldots, x_n) = \frac{P(z, t_1, \ldots, t_n)}{Q(z, t_1, \ldots, t_n)}\big|_{z=F(x_1,\ldots,x_n), t_k = x_1^k + x_2^k + \ldots + x_n^k} \tag{2}$$

Proof:

Expose an explicit formula: Consider the following fraction:

$$U(z, x_1, \ldots, x_n) = \frac{G(x_1, \ldots, x_n)}{z - F(x_1, \ldots, x_n)} \tag{3}$$

Take the average with respect of an action of all permutations of symmetric group S_n on indeterminates $x_1, x_2, \ldots, x_n$:

$$U_{\text{average}}(z, x_1, \ldots, x_n) = \frac{1}{n!} \sum_{\sigma \in S_n} U^\sigma(z, x_1, \ldots, x_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \frac{G^\sigma(x_1, \ldots, x_n)}{z - F^\sigma(x_1, \ldots, x_n)}$$

Because of symmetrization $U_{\text{average}}(z, x_1, \ldots, x_n)$ is a fraction such that its nominator and denominator can be considered as polynomials over symmetric polynomials $t_1, t_2, \ldots, t_n$:

$$U_{\text{average}}(z, x_1, \ldots, x_n) = \frac{P(z, t_1, \ldots, t_n)}{Q(z, t_1, \ldots, t_n)}\big|_{t_k = x_1^k + \ldots + x_n^k}. \tag{4}$$

where

$$Q\left(z, t_1, \ldots, t_n\right)\big|_{t_k = x_1^k + \ldots + x_n^k} = \prod_{r=1}^{l} \left(z - F^{\sigma_r}(x_1, \ldots, x_k)\right)\,, \tag{5}$$

$\{\sigma_1, \dots \sigma_l\}, (l = \frac{n!}{|H_F|})$ are any representatives of left equivalence classes of the group S_n with respect to the subgroup H_F *.

One can see that

$$G(x_1, \dots, x_n) = \frac{n!}{|H_F|} Res \left(\frac{P(z, t_1, \dots, t_n)}{Q(z, t_1, \dots, t_n)} \right) |_{z=F(x_1,\dots,x_n), t_k = x_1^k + x_2^k + \dots + x_n^k} = \quad (6)$$

$$\frac{n!}{|H_F|} \left(\frac{P(z, t_1, \dots, t_n)}{Q'(z, t_1, \dots, t_n)} \right) |_{z=F(x_1,\dots,x_n), t_k = x_1^k + x_2^k + \dots + x_n^k} \quad (7)$$

This is just what we claimed in (2). To prove this relation we note that

1) every permutation $\sigma \in S_n$ preserves polynomial G if it preserves polynomial F:

$$F^\sigma = F \Rightarrow G^\sigma = G, \quad \text{because} \quad H_F \leq H_G$$

2) Residue of the function $\frac{1}{z - F^\sigma}$ in the point $z = F$ is equal to one if $\sigma \in H_F$ and it is equal to zero if $\sigma \notin H_F$:

$$Res \left(\frac{1}{z - F^\sigma} \right) |_{z=F} = \begin{cases} 1 & \text{if} \sigma \in H_F \\ 0 & \text{if} \sigma \notin H_F \end{cases}$$

Hence

$$\frac{n!}{|H_F|} Res \left(\frac{P(z, t_1, \dots, t_n)}{Q(z, t_1, \dots, t_n)} \right) |_{z=F(x_1,\dots,x_n), t_k = x_1^k + x_2^k + \dots + x_n^k}$$

$$\frac{1}{|H_F|} Res \left(\sum_{\sigma \in S_n} \frac{G^\sigma(x_1, \dots, x_n)}{z - F^\sigma(x_1, \dots, x_n)} \right) |_{z=F(x_1,\dots,x_n)} =$$

$$\frac{1}{|H_F|} \sum_{\sigma \in H_F} G^\sigma(x_1, \dots, x_n) = \frac{1}{|H_F|} \sum_{\sigma \in H_F} G(x_1, \dots, x_n) = G(x_1, \dots, x_n) \blacksquare$$

Example

Consider

$$F = x_1^2 + x_2^2, G = x_3$$

Then $H_F = H_G$. Express $G = x_3$ via $F = x_1^2 + x_3^2$ and t_1, t_2, t_3. According to (2-7):

$$U_{\text{average}} = \frac{1}{6} \sum_{\sigma \in S_3} \frac{x_3^\sigma}{z - (x_1^2 + x_2^2)^\sigma} =$$

$$\frac{1}{3} \left(\frac{x_3}{z - (x_1^2 + x_2^2)} + \frac{x_2}{z - (x_1^2 + x_3^2)} + \frac{x_1}{z - (x_2^2 + x_3^2)} \right) =$$

* Transformations $\sigma \in H_F$ do not change denominator $z - F(x_1, \dots, x_n)$ of the fraction. Hence for every $g \in S_n$, $g = h\sigma_r$, where $h \in GH_F$ and $z - F^{h\sigma_r} = z - F^{\sigma_r}$

For simplicity assume that x_1, x_2, x_3 are roots of the equation $x^3 + x - 1 = 0$, i.e. $t_1 = x_1 + x_2 + x_3 = 0$, $t_2 = x_1^2 + x_2^2 + x_3^2 = -2$ and $t_3 = x_1^3 + x_2^3 + x_3^3 = 3$ (In this case we have simplification: $x_1^2 + x_2^2 = -x_3^2 - 2$ but still calculations are not very quick)

$$U_{\text{average}} = \frac{1}{3}\left(\frac{x_3}{z + (x_3^2 + 2)} + \frac{x_2}{z + (x_2^2 + 2)} + \frac{x_1}{z + (x_1^2 + 2)}\right) = \frac{-3z - 5}{z^3 + 4z^2 + 5z + 3}$$

and according to (7)

$$x_3 = Res\left(\frac{-3z - 5}{z^3 + 4z^2 + 5z + 3}\right)\Big|_{z = x_1^2 + x_2^2} = \left(\frac{-3z - 5}{3z^2 + 8z + 5}\right)\Big|_{z = x_1^2 + x_2^2} =$$

$$\frac{-3(x_1^2 + x_2^2) - 5}{3(x_1^2 + x_2^2)^2 + 8(x_1^2 + x_2^2) + 5}$$

One can see that indeed it is a fact.