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Orthocentre of triangle and related problems

Three heights (altitudes) of triangle intersect at the point. This is well-known statement *

I remember how I was surprised in school when I realised that this happens since orthocentre of $\triangle ABC$ coincides with centre of circumstribed circle of 'double' triangle $\triangle A'B'C' = 2 \times \triangle ABC$!

Remark We will use notation $\triangle A'B'C' = 2 \times \triangle ABC$ for triangle such that

$$\begin{array}{c} \text{side } A'B' \text{ passes via the point } C \text{ and } A'B'||AB \\ \text{side } B'C' \text{ passes via the point } A \text{ and } B'C'||BC \\ \text{side } A'C' \text{ passes via the point } B \text{ and } A'C'||AC \end{array} \qquad (1)$$

We consider another remarkable point of $\triangle ABC$:

$$\text{intersection of heights (altitudes) of double triangle} \triangle A'B'C' = 2 \times \triangle ABC \qquad (2)$$

or in other words the centre of circumscribed circle of the 'quatre' triangle $\triangle \tilde{A}\tilde{B}\tilde{C} = 2 \times \triangle A'B'C'$, where $\triangle A'B'C' = 2 \times \triangle ABC$.

Many years ago I was solving the following problem: Let $ABCD$ be tetraedron such that all its faces are four equal triangles (with sides a, b, c). Calculate its volume** At that time I came to the following very beautiful solution of this problem:

Consider right parallelipiped $ABCDA'B'C'D'$ with sides x, y, z such that this tetrahedron is inscribed in this parallelepiped:

$$\begin{cases} AD = BC = A'D' = B'C' = x \\ AB = CD = A'B' = C'D' = y \\ AA" = BB' = CC" = DD" = z \end{cases}, \quad \text{where} \begin{cases} x^2 + y^2 = a^2 \\ y^2 + z^2 = b^2 \\ z^2 + x^2 = c^2 \end{cases} \quad i.e. \begin{cases} x = \sqrt{\frac{a^2+c^2-b^2}{2}} \\ y = \sqrt{\frac{b^2+a^2-c^2}{2}} \\ z = \sqrt{\frac{b^2+c^2-a^2}{2}} \end{cases} \qquad (3)$$

We see that triangle which forms tetrahedron has to be acute (not obtuse), since x, y, z have to be positive (or non zero)

Now we see that volume of our tetrahedron is equal to

$$\mathrm{Vol}\,(AB'CD') = \mathrm{Vol}\,(ABCDA'B'C'D') - \mathrm{Vol}\,(BACB') - \mathrm{Vol}\,(C'B'D'C) -$$

* Arnold makes it famous claiming that this happens due to Jacobi identity. (see the etude in my homepage) However we will speak here about other topic.

** This problem comes from 1984 when I was tutoring Vahagn Minasian... `Ou es-tu maintenant, Vahagn?`

$$\mathrm{Vol}\,(DACD') - \mathrm{Vol}\,(A'B'D'A) = xyz - \frac{xyz}{6} \cdot 4 = \frac{xyz}{3} =$$

$$\frac{\sqrt{2}}{12}\sqrt{(a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)} \tag{4}$$

Yes, this is beautiful. However there is also another solution. Thirty years ago trying to construct this tetrahedron, I came the equations

$$\begin{cases} {a'}^2 + h^2 = a^2 \\ {b'}^2 + h^2 = b^2 \\ {c'}^2 + h^2 = c^2 \end{cases} . \tag{5}$$

Here a, b, c are edges of the $\triangle ABC$, $AB = c, BC = a$ and $AC = b$. For an arbitrary point P on the plane denote by $x_a = PA$, $x_b = PB$ and $x_c = PC$. If you find a point P such that these equations are fullfilled then,

$$\mathrm{Vol}\,(tetrahedron) = \frac{h \cdot \text{Area of the} \triangle ABC}{3} = \tag{6}$$

I could not solve these equations thirty years ago. A week ago I told this problem to my friend, Hovik Nersessian. He suggested that a point P is related with orthocentre... This solves the puzzle! I realised that the following statements are obeyed:

Theorem

Let ABC be a triangle on the plane with edges, $a = BC$, $b = AC$ and $c = AB$. For every point P on the plane consider

$$\begin{cases} x_a = PA = \textit{the length of the segment from point } P \textit{ to the point } A \\ x_b = PB = \textit{the length of the segment from point } P \textit{ to the point } B \\ x_c = PC = \textit{the length of the segment from point } P \textit{ to the point } C \end{cases} . \tag{7}$$

Then

1) there exist a unique point $P = M_1$ such that at this point

$$\begin{cases} x_a^2 - x_b^2 = b^2 - a^2 \\ x_b^2 - x_c^2 = c^2 - b^2 \\ x_c - x_a^2 = a^2 - c^2 \end{cases} \tag{8}$$

and this point is orthocentre of the triangle ABC.

2) there exist a unique point $P = M_2$ such that at this point

$$\begin{cases} x_a^2 - x_b^2 = a^2 - b^2 \\ x_b^2 - x_c^2 = b^2 - c^2 \\ x_a - x_c^2 = a^2 - c^2 \end{cases} \tag{9}$$

and this point is orthocentre of the double triangle triangle ABC.

(`Note, please that RHS in equations (8) and (9) differ by sign!`)

3) the transformation

$$\begin{cases} x_a^2 \mapsto 2x_b^2 + 2x_c^2 - a^2 \\ x_b^2 \mapsto 2x_a^2 + 2x_c^2 - b^2 \\ x_c^2 \mapsto 2x_a^2 + 2x_b^2 - c^2 \end{cases} \qquad (10)$$

transforms solutions of the equation (8) to solutions of equation (9); and vice versa

$$\begin{cases} x_a^2 \mapsto \frac{x_b^2+x_c^2-x_a^2+b^2+c^2-a^2}{4} \\ x_b^2 \mapsto \frac{x_a^2+x_c^2-x_b^2+a^2+c^2-b^2}{4} \\ x_c^2 \mapsto \frac{x_a^2+x_b^2-x_c^2+a^2+b^2-c^2}{4} \end{cases} \qquad (11)$$

transforms solutions of the equation (9) to solutions of equation (8)†*;*

4) If $DABC$ is tetrahedron such that all its faces are equal to the triangle ABC, then it is acute triangle, if M_2 is the pointin $\triangle ABC$ which is the orthocentre of the double triangle, then DM_2 is the height. The proof is almost evident.

Notice that points on the line which passes via the point C and is orthogonal to the edge AB is the locus of the points M such that

$$d(M,A)^2 - d(M,B)^2 = b^2 - a^2\,.$$

We see that points on the height obey this property. The same is true for the line orthogonal to BC which passes through the vertex A and for the line orthogonal to AC which passes via vertex B.

Thus we see that three heights of triangle intersect at the point and equation (8) is obeyed.

Equation (9) looks almost the same, but...!!! here we come to a different point.

Points on the line which passes via the point C' (vertex of the double triangel) and is orthogonal to the edge AB is the locus of the points M such that

$$d(M,A')^2 - d(M,B')^2 = a^2 - b^2\,,$$

and respectively points on the line which passes via the point B and is orthogonal to the edge AC is the locus of the points M such that

$$d(M,A)^2 - d(M,C)^2 = a^2 - c^2\,,$$

and respectively points on the line which passes throug the point A and is orthogonal to the edge BC is the locus of the points M such that

$$d(M,B)^2 - d(M,C)^2 = b^2 - c^2\,.$$

† *One can see that formal transformation $x_a \mapsto ix_a$ $x_b \mapsto ix_b$ $x_c \mapsto ix_c$ does this!*

These equations imply that M_2 is the intersection point of heights of double trianfgle.

Now about symmetry between equations (8) and (9)

It is convenient to denote by x_a, x_b, x_c solutions of equation (8) and by y_a, y_b, y_c solutions of equation (9).

First consider transformation (10) Consider double triangle $A'B'C'$ and triangle $C'M_2B'$, where M_2 is orthocentre of the double triangle. We see that $M_2A = y_a$. Point M_2 is orthocentre of the big triangle. Hence by similarity with edges $C'M_2 = 2x_c$ and $B'M_2 = 2x_b$, and y_a is median. We come to

$$y_a^2 = 2x_b^2 + 2x_c^2 - a^2\,.$$

This proves (10). To prove transformation (11) consider in triangle ABC half-triangle $A'B'C'$. We will come to (11).