On one triple

Let M be a (super)manifold, and $\mathcal{M} = C^\infty(M)$ be an algebra of functions on this manifold. Consider the triple (1):

Poisson bracket on $C^\infty(M) \to$ Homotopy Poisson bracket on $C^\infty(M) \to$ Vector field on $C^\infty(M)$

Where it comes from?:

Let H be an arbitrary even Hamiltonian on ΠT^*M (arbitrary odd Hamiltonian on T^*M). It defines homotopy even Poisson bracket on M, the series of bracets: for every n

$$\{f_1, \dots, f_n\} = [\dots [H, f_1], f_2] \dots, f_n]\big|_M \,, \tag{2a}$$

where $[\,,\,,\,]$ is canonical odd Poisson bracket on ΠT^*M. (Respectively in the case if H is an odd Hamiltonian on T^*M we come to homotopy odd Poisson bracket

$$\{f_1, \dots, f_n\} = (\dots (H, f_1), f_2) \dots, f_n)_M \,, \tag{2b}$$

where $(\,,\,,\,)$ is canonical even Poisson bracket on ΠT^*M.)

The Jacobi identity for the series of homotopy even Poisson brackets (2a) is provided with master-equation

$$[H, H] = 0 \tag{3a}$$

(Respectively the Jacobi identity for the series of homotopy odd Poisson brackets (2b) is provided with master-equation

$$(H, H) = 0 \tag{3b}$$

) If Hamiltonian H is quadratic in fibers, then we come to usual Poisson bracket. If H is an arbitrary function we come to homotopy Poisson bracket, and if H is the linear over fibers we come to vector field.

We see that usual Poisson bracket and vector field are related with each other being the special cases of homotopy Poisson bracket. All three structures are represented by homological vector field on infinite-dimensional manifold $\mathcal{M} = C^\infty(M)$

$$f \mapsto f + \tau H\left(x, \frac{\partial f}{\partial x}\right), Q = \int dx H\left(x, \frac{\delta f}{\delta x}\right)\frac{\delta}{\delta x} \tag{4}$$

In the case of linear Hamiltonian vector field (4) is just the image of vector field generated by the linear Hamiltonian:

$$x^i \mapsto x^i + \tau K^i \longrightarrow f \mapsto f(x^i + \tau K^i)$$

`I cannot avoid the temptation to compare this triple whith another triple:`

Many years ago I learned from the book of Polia about the triple

$$\text{square} \to \text{polygon} \to \text{trianlge}$$

it can be used for the very elegant proof of Pythagorean Theorem:

Let $\triangle ACB$ be rectangular triangle, $\angle C = \frac{\pi}{2}$. We denote $a = |BC|, b = |AC|, c = |AB|$

Consider the following three statements

I. Let $\alpha_4, \beta_4, \gamma_4$ be three squares such that the square α_4 is the square on the side BC, i.e. its side is equal to a, the square β_4 is the square on the side AC, i.e. its side is equal to b, and the square γ_4 is the square on the side AB, i.e. its side is equal to c, then

$$a^2 + b^2 = c^2 , \qquad \text{Pythagorean Theorem}$$

which is nothing but the statement

$$\text{Area of the square } \alpha_4 + \text{Area of the square } \beta_4 = \text{Area of the square } \gamma_4 \qquad (*)$$

II. Consider three similar n-gones, (polygones with n sides), $\alpha_n, \beta_n, \gamma_n$, such that one of the sights of polygon α_n is equal to a, the corresponding side of the similar polygon β_n is equal to b and the corresponding side of the similar polygon γ_n is equal to c.

Then

$$\text{Area of the polygon } \alpha_n + \text{Area of the polygon } \beta_n = \text{Area of the polygon} \gamma_n \qquad (**)$$

III. Consider three similar rectangular α_3, β_3 and γ_3, $\triangle\alpha_3 \sim \triangle\beta_3 \sim \triangle\gamma_3$ such that hypothenuse of the triangle α_3 is equal to a, hypothenuse of the triangle β_3 is equal to b, and hypothenuse of the triangle γ_3 is equal to c then

$$\text{Area of } \triangle\alpha_3 + \text{Area of } \triangle\beta_n = \text{Area of } \triangle\gamma_3 \qquad (***)$$

The statements I, II and III are equivalent. since Area of every polygon is proportional to the square of the side. This means that prove the Pythagoeran Teorem it suffices to prove just one of these statements. Prove the statement (***).

Let CD be the height of the triangle ACB, $CD \perp AB$ and $D \in AB$.

Triangles ACD and BCD are rectangular triangles, they are similar to the triangle ABC. Hypothenuse of triangle ACD is equal to a hypothenuse of triangle BCD is equal to b Hypothenuse of triangle ACB is equal to c. We have that

$$\text{Area}(\triangle ACD) + \text{Area}(\triangle BCD) = \text{Area}(\triangle ACB)$$

This proves (***). This proves Pythagorean Theorem.

To prove this theorem we first generalise the statement from suqre to arbitrary polygon, then specify the special type of polygon--- rectangular triangle.