A Tale on Differential Geometry

Once upon a time there was an ant living on a sphere of radius R. One day he asked himself some questions: What is the structure of the Universe (surface) where he lives? Is it a sphere? Is it a torus? Or may be something more sophisticated, e.g. pretzel (a surface with two holes)

Three-dimensional human beings do not need to be mathematicians to distinguish between a sphere torus or pretzel. They just have to look on the surface. But the ant living on two-dimensional surface cannot fly. He cannot look on the surface from outside. How can he judge about what surface he lives on *?

Our ant loved mathematics and in particular *Differential Geometry*. He liked to draw various triangles, calculate their angles α, β, γ, area $S(\Delta)$. He knew from geometry books that the sum of the angles of a triangle equals π, but for triangles which he drew it was not right!!!!

Finally he understood that the following formula is true: For every triangle

$$\frac{(\alpha + \beta + \gamma - \pi)}{S(\Delta)} = c \tag{1}$$

A constant in the right hand side depended neither on size of triangle nor the triangles location. After hard research he came to conclusion that its Universe can be considered as a sphere embedded in three-dimensional Euclidean space and a constant c is related with radius of this sphere by the relation

$$c = \frac{1}{R^2} \tag{2}$$

...Centuries passed. Men have deformed the sphere of our old ant. They smashed it. It seized to be round, but the ant civilisation survived. Moreover old books survived. New ant mathematicians try to understand the structure of their Universe. They see that formula (1) of the Ancient Ant mathematician is not true. For triangles at different places the right hand side of the formula above is different. Why? If ants could fly and look on the surface from the cosmos they could see how much the sphere has been damaged by humans beings, how much it has been deformed, But the ants cannot fly. On the other hand they adore mathematics and in particular *Differential Geometry*. One day considering for every point very small triangles they introduce so called curvature for every point P as a limit of right hand side of the formula (1) for small triangles:

$$K(P) = \lim_{S(\Delta) \to 0} \frac{(\alpha + \beta + \gamma - \pi)}{S(\Delta)}$$

Ants realise that curvature which can be calculated in every point gives a way to decide where they live on sphere, torus, pretzel... They come to following formula ** : integral of curvature over the whole Universe (the sphere) has to equal 4π, for torus it must equal 0, for pretzel it equalts -4π...

$$\frac{1}{2\pi} \int K(P)dP = 2\,(1 - \text{number of holes})$$

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* This is not very far from reality: For us human beings it is impossible to have a global look on three-dimensional manifold. We need to develop local methods to understand global properties of our Universe. *Differential Geometry* allows to study global properties of manifold with local tools.

** In human civilisation this formula is called Gauß-Bonet formula. The right hand side of this formula is called Euler characteristics of the surface.