On one property of quadrics.

4 XII 2013

About two months ago Gabor Megyesi suggested me the following problem:

Let C be a quadric in the plane. Denote by C^* a locus of the points ~~such~~ in $\mathbb{E}^3$ such that any of these points is a vertex of circular cone over the curve C. (E.g. if C is a circle then C^* is a line ~~which is orthogonal~~ passing through the centre of this circle and orthogonal to the plane of circle).

Show that in the case if C is not a circle then C^* is a quadric too, and

$$(C^*)^* = C. \qquad (1)$$

I solved the problem, and my answer (seems to be) ~~is~~ beautiful. On the other hand my solution in some sense is "brute force" solution. I still ~~cannot find~~ have not more beautiful (and illuminating) solution.

Here I will state an answer and explain ~~my~~ calculations.

First I formulated detailed answer:

Statement. Let C be a quadric in the plane α with foci F_1, F_2. Then C^* is a quadric in the plane β wich is orthogonal to plane α and intersects with α by the line F_1F_2. The quadric C^* passes via foci F_1 and F_2. Respectively the quadric C passes via foci of curve C^*.

~~If~~ Let $\vec{r}$, $\vec{R}$ be arbitrary points on curves C, C^* respectively, and $\vec{v}$, $\vec{V}$ be tangent velors. ($\vec{v}$ at a point $\vec{r}$ of curve C, and $\vec{V}$ at a point $\vec{R}$ of a curve C^*) Then:

$\vec{v}$ is directed along the axis of circular cone over C^*

$\vec{V}$ is directed along the axis of circular cone over C

and the following relation holds.

$$(\vec{r}-\vec{R}, \vec{r}-\vec{R})(\vec{v}, \vec{V}) = (\vec{r}-\vec{R}, \vec{v})(\vec{r}-\vec{R}, \vec{V}) \quad (2)$$

If $C: \frac{x^2}{A} + \frac{y^2}{B} = 1$. $(A>0,\ B<A)$, $z=0$,

then $C^*: \frac{x^2}{A-B} - \frac{z^2}{B} = 1$, $y=0$. (3)

$(\text{ellipse})^* = \text{hyperbola}$, $(\text{hyperbola})^* = \text{ellipse}$.

In the degenerate case (C is parabola) answer is ~~analogous~~. the same:
C^* is a parabola passing through Focus F ~~If~~ of parabola C.

If C: ~~$y = px^2$~~ $x = py^2$, $z = 0$

C^*: $x = \frac{1}{4p} - pz^2$, $y = 0$

Now we will prove this statement.
I hope my calculations are not too long.
[I accept that they are not illuminating.]

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Denote as in (2) by $\vec{F}$ an arbitrary point on quadric C and by $\vec{R}$ an arbitrary point on a curve C^*. [We still do not know (still) that is a quadric]. Let $\vec{M}$ be a vector attached at the point $\vec{R}$ such that ~~$\vec{M}$~~ is directed along axis of circular cone over curve C.

The condition that $\vec{R}$ is a vertex of circular cone and $\vec{M}$ is directed along axis means that for arbitrary $F \in C$

$$\delta \frac{\vec{F} - \vec{R}}{|\vec{F} - \vec{R}|} \text{ is orthogonal to } \vec{M} \qquad (3a)$$

(where δ (a variation) is along curve C).

The condition (3a) means that

$$\left(\left[(\vec{r}-\vec{R})\times\left[(\vec{r}-\vec{R})\times\vec{V}\right]\right],\vec{M}\right)=0 \qquad (3b)$$

where $\vec{V}$ is tangent vector to curve C ~~t~~ is at the point $\vec{r}$ ($\vec{V}$ is proportional to velocity vector) Here $\times$ is vector product and $(,)$ - scaler product.

One can rewrite (3a) as

$$(\vec{r}-\vec{R},\vec{r}-\vec{R})(\vec{V},\vec{M})\equiv(\vec{r}-\vec{R},\vec{V})(\vec{r}-\vec{R},\vec{M}) \qquad (3c)$$

~~The~~ For given ~~pot~~ $\vec{R}$ (vertex of cone over C) and $\vec{M}$ (vector which defines axis) this relation holds for arbitrary point $\vec{r}$ and arbitrary tangent vector $\vec{V}$ at $\vec{r}$.

We will prove now that C^* is indeed a quadric which obeys conditions above, and vector $\vec{M}$ is tangent to quadric C^*.

Perform calculations.

Choose (Cartesian) coordinates such that quadric C is defined by equation

$$\frac{x^2}{A}+\frac{y^2}{B}=1,\ z=0 \qquad (4)$$

$$A>0,\quad A>B.$$

If $B>0$ then C is ellipse. If $B<0$ then C is hyperbola

We denote $\vec r = (x, y, z)$ and $\vec R = (X, Y, Z)$

Now we solve eq. (3a). Recall that for given $\vec R$ and $\vec M$ it is obeyed for arbitrary $\vec r$ on the curve C and arbitrary $\vec v$ which is tangent to C, at $\vec r$.

It is evident by symmetry arguments that

$$\vec R = (X, Y, Z) = (X, 0, Z) \quad [Y \equiv 0)$$

and

$$\vec M = (M_x, M_y, M_z) = (M_x, 0, M_z), \quad M_y \equiv 0$$

Equation (3c) can be rewritten

$$[(x-X)^2 + y^2 + Z^2]\, v_x M_x =$$

$$= [(x-X)v_x + y v_y][(x-X)M_x - Z M_z] \qquad (5)$$

Since is defined by eq. (4) we have that

$$\frac{x\,\delta x}{A} + \frac{y\,\delta y}{B} = 0, \text{ i.e. } \vec v \sim \left[\frac{y}{B}, -\frac{x}{A}\right]$$

We can put in (5) $\vec v = \left(\frac{y}{B}, -\frac{x}{A}, 0\right)$. Hence Opening brackets and dividing on y we come to.

$$(y^2+Z^2)\frac{M_x}{B} + M_x(x-X)\frac{x}{A} + M_z Z(x-X)\frac{1}{B} - M_z\frac{Zx}{A} = 0.$$

Using (4) we come

$$M_x\left(1+\frac{Z^2}{B}\right) - M_z\frac{ZX}{B} +$$

$$+ \left[-M_x\frac{x}{A} + M_z Z\left(\frac{1}{B} - \frac{1}{A}\right)\right] x \equiv 0.$$

This relation holds for arbitrary $\underline{x}$.

$$\Downarrow$$

$$\begin{cases} \left(1+\frac{Z^2}{B}\right) M_x - \frac{ZX}{B} M_y = 0 \\ -\frac{X}{A} M_x + Z\left(\frac{1}{B}-\frac{1}{A}\right) M_y = 0. \end{cases}$$

$$\Downarrow$$

$$\det \begin{pmatrix} 1+Z^2/B, & -\frac{ZX}{B} \\ -\frac{X}{A}, & Z\left(\frac{1}{B}-\frac{1}{A}\right) \end{pmatrix} = 0$$

$$\left(1+\frac{Z^2}{B}\right)\left(\frac{1}{B}-\frac{1}{A}\right) = \frac{X^2}{AB}$$

$$\frac{X^2}{A-B} - \frac{Z^2}{B} = 1,$$

$$\vec{M} = [M_x : 0 : M_y] = \left[\frac{ZX}{B} : 0 : 1+\frac{Z^2}{B}\right] =$$

$$= \left[\frac{ZX}{B} : 0 : \frac{X^2}{A-B}\right] = \left[\frac{Z}{B} : 0 : \frac{X}{A-B}\right]$$

We see that C^* is quadric passing through foci of C and vector

$\vec{M}$ is directed along tangent

vector

4 XII 2013