Connection on manifold.

§1 General construction

Show an existence of globally defined connection on (paracompact) manifold.

Let M be a manifold with a smooth atlas $\mathbf{A} = \{U_\alpha, \varphi_\alpha\}$ and with partition of unity $\{\rho_\alpha\}$ subordinate to this atlas.

One can define in any open domain U_a connection ${}^{(\alpha)}\Gamma$ such that components of Christoffel indices are equal to zero in the coordinates $\{x^i_\alpha\}$. Denote by ${}^{(\alpha)}_{(\beta)}\Gamma^i_{km}$ components of Christoffel tensor ${}^{(\alpha)}\Gamma$ in coordinates $\{x^i_\beta\}$ (in the coordinate domain U_α). (Sure components ${}^{(\alpha)}_{(\beta)}\Gamma^i_{km}$ are defined in intersection $U_\alpha \cap U_\beta$ of coordinate domains U_α and U_β.) Transformation law for any connection from coordinates $\{x^i_\beta\}$ to coordinates $\{x^i_{\beta'}\}$ are the the following

$$ {}^{(\alpha)}_{(\beta')}\Gamma^{i'}_{k'm'} = \frac{\partial x^{i'}_{(\beta')}(\mathbf{x})}{\partial x^i_{(\beta)}} \frac{\partial x^k_{(\beta)}}{\partial x^{k'}_{(\beta')}} \frac{\partial x^m_{(\beta)}}{\partial x^{m'}_{(\beta')}} {}^{(\alpha)}_{(\beta)}\Gamma^i_{km}(\mathbf{x}) + \frac{\partial x^{i'}_{(\beta')}}{\partial x^i_{(\beta)}} \frac{\partial^2 x^i_{(\beta)}(\mathbf{x})}{\partial x^{k'}_{(\beta')} \partial x^{m'}_{(\beta')}} \,. \tag{1} $$

Remark The second term in transformation law *does not depend* connection. It depends only on coordinates $x_{(\beta)}$ and coordinates $x_{(\beta')}$.

In particular for the local connection in U_α which has vanishing Christoffel symbol in coordinates $\{x_{(\alpha)}\}$ we have that ${}^{(\alpha)}_{(\alpha)}\Gamma^i_{km} = 0$ and for any $x \in U_\alpha \cap U_\beta$

$$ {}^{(\alpha)}_{(\beta)}\Gamma^i_{km}(\mathbf{x}) = \frac{\partial x^i_{(\beta)}}{\partial x^{i'}_{(\alpha)}} \frac{\partial^2 x^{i'}_{(\alpha)}(\mathbf{x})}{\partial x^k_{(\beta)} \partial x^m_{(\beta)}} \,. \tag{2} $$

This formula defines the Christoffel symbols for local connection ${}^\alpha\Gamma$ in coordinates x_β.

Now use the partition of unity $\{\rho_\alpha\}$ subordinate to the covering U_α and consider

$$ {}_{(\beta)}\Gamma^i_{km}(\mathbf{x}) = \sum_\alpha \rho_\alpha(\mathbf{x})\, {}^{(\alpha)}_{(\beta)}\Gamma^i_{km}(\mathbf{x}) = \sum_\alpha \rho_\alpha(\mathbf{x}) \frac{\partial x^i_{(\beta)}}{\partial x^{i'}_{(\alpha)}} \frac{\partial^2 x^{i'}_{(\alpha)}(\mathbf{x})}{\partial x^k_{(\beta)} \partial x^m_{(\beta)}} \,. \tag{3} $$

Proposition The formula (3) defines globally a smooth connection on M (in coordinates $\{x^i_{(\beta)}\}$)

The definition of partition of unity implies that for any set of indices (i,k,m) the formula above defines the globally defined smooth function for an arbitrary indices (i,k,m). One have to prove that at any point $\mathbf{x}$ the entries $\Gamma(\mathbf{x})$ transform as necessary.

To show that the formula (3) defines the connection it suffices to show that the difference between the expression (1.3) and an arbitrary connection in a vicinity of an arbitrary point transforms as a tensor field of valency (1.2). Let $\mathbf{x} \in U_\beta \cap U_\gamma$. Consider in a vicinity of a point $\mathbf{x}$ locally defined connection ${}^{(\beta)}\Gamma$. Recall that a locally defined connection ${}^{(\beta)}\Gamma$ has vanishing Christoffel symbols in coordinates x^i_β. One can see that for an arbitrary coordinates $\{x^i_\gamma\}$

$$ {}_{(\gamma)}\Gamma^i_{km}(\mathbf{x}) - {}^{(\beta)}_{(\gamma)}\Gamma(\mathbf{x}) = \sum_\alpha \rho_\alpha(\mathbf{x})\, {}^{(\alpha)}_{(\gamma)}\Gamma^i_{km}(\mathbf{x}) - \sum_\alpha \rho_\alpha(\mathbf{x})\, {}^{(\beta)}_{(\gamma)}\Gamma^i_{km}(\mathbf{x}) = \sum_\alpha \rho_\alpha(\mathbf{x}) \left({}^{(\alpha)}_{(\gamma)}\Gamma^i_{km}(\mathbf{x}) - {}^{(\beta)}_{(\gamma)}\Gamma^i_{km}(\mathbf{x}) \right) $$

It follows from the formula (1) that under changing of coordinates from $x^i_{(\gamma)}$ to coordinates $x^i_{(\gamma')}$ the difference will change as a tensor field.

Remark The connection that we constructed is "glued"" through local flat connections with use of partition of unity. Of course in general it is not flat connection. Study examples...

§2 An example

We consider as an example a sphere S^2 with different partitions of unity.

On the sphere $x^2 + y^2 = z^2 = a^2$ of the radius a one can consider stereographic coordinates $(u+, v_+)$ (stereographic projection from the sphere on the plane $z = 0$ with respect to the South Pole) and (u_-, v_-) (stereographic projection from the sphere on the plane $z = 0$ with respect to the North Pole) We have