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On Duistermaat-Heckman localisation Theorem II

Here we will give a formulation (with supermathematics flavor), the proof and concrete calculations for DH (Dustermaat-Heckman) localisation formula. This etude is essentially based on the papers of Armen Nersessian [1], and of Oleg Zaboronsky and Albert Schwarz [2], my etude [4] (see the previous etude on this topic) which was based on calculations of A.Belavin.) It is interesting also to note the paper [3]. This etude is a developed exposition of my talk on the Geometry seminars in Manchester (17 October and 23 October, 2013).

If a form, is invariant with respect to odd vector field
$Q = d + \iota_{\mathbf{K}} = \sqrt{\mathcal{L}_{\mathbf{K}}}$ where $\mathcal{L}_{\mathbf{K}}$ is Lie derivative
with respect to $U(1)$-vector field $\mathbf{K}$, then integral of this
form over manifold M is localised at the zero locus
of vector fied K. This is the meaning.
of Dustermaat-Heckman localisation formula.

§0 Recallings

Recall briefly the DH (Duistermaat-Heckman) localisation formula and perform some calculations based on calculations in [4].

Let (M, Ω) be compact symplectic supermanifold (Ω is non-degenerate closed two form, $\dim M = 2n$). Let H be a Hamiltonian and $\mathbf{K} = D_H$: $dH = -\iota_{\mathbf{K}}\Omega$, its Hamiltonian vector field. Let vector field K obeys the following conditions:

$$\mathbf{K} = D_H \text{ is compact vector field, i.e. it defines } U(1)\text{-action on } M^{1)} \tag{0.1}$$

$$\text{Zero locus of vector field } \mathbf{K},\ \mathbf{K}(x_i) = 0, \text{ is a set } \{x_i\} \text{ of isolated points} \tag{0.2}$$

DH-localisation formula states that if conditions (0.1) and (0.2) are obeyed then

$$\int e^{iH} dV_\Omega = \int e^{i(H+\Omega)} = \sum_{x_i} \frac{e^{iH}\sqrt{\det \Omega_{ik}}}{\sqrt{\det \operatorname{Hess} H}}\Big|_{x_i} = \sum_{x_i} \frac{e^{iH(x_i)}}{\sqrt{\det\left(\frac{\partial K(x)}{\partial x}\big|_{x=x_i}\right)}}\,. \tag{0.3}$$

Comments to this formula:

1. Here and later we often omit all the coefficients proportional to $\pi^a, n!, i^n$,

2. x_i: $\mathbf{K}(x_i) = 0$, is a locus (zero locus) of Hamiltonian vector field $\mathbf{K}$, i.e. stationary points of Hamiltonian H,

3. dV_Ω is invariant volume form:

$$dV_\Omega = \Omega^n = \underbrace{\Omega \wedge \ldots \wedge \Omega}_{n\text{-times}} \text{ is Lioville volume form}\,,$$

in local coordinates $dV_\Omega = \mathrm{Pfaf}\,\Omega d^{2n}x = \sqrt{\det\Omega}d^{2n}x$, $\mathrm{Hess}\,H = \frac{\partial^2 H}{\partial x^i \partial x^k}$ is bilinear form at stationary points; as well as $\frac{\partial K}{\partial x}$ is linear operator at zero locus of vector field $\mathbf{K}$.

Shortly show how to calculate (0.3) using ideas of [4].

Let ω be an arbitrary $\mathbf{K}$-invariant 1-form:

$$\mathcal{L}_{\mathbf{K}}\omega = d\circ\iota_{\mathbf{K}}\omega + \iota_{\mathbf{K}}\circ d\omega = 0\,. \tag{0.4}$$

Consider 'partition function

$$Z(t) = \int_M e^{i((H+\Omega)+td_{\mathbf{K}}\omega)}\,, \tag{0.5}$$

where $d_{\mathbf{K}} = d + \iota_{\mathbf{K}}$. One can see that condition (0.4) and condition $d_{\mathbf{K}}(H+\Omega) = 0$ imply that this partition function does not depend on t:

$$\frac{dZ(t)}{dt} = i\int_M d_K\left(\omega e^{i((H+\Omega)+td_{\mathbf{K}}\omega)}\right) = 0\,, \tag{0.6}$$

because for an arbitrary differential form F, $\int_M dF = 0$ (Stokes Theorem) and $\int_M \iota_{\mathbf{K}}F = 0$ also, since form $\iota_{\mathbf{K}}F$ has order less than equal $2n$ ($2n$ is the dimension of M is an order of top form.)

Partition function $Z(t)$ at $t = 0$ is the left hand side of equation (0.3), the initial integral; this function at $t\to\infty$ can be calculated using stationary phase method. So using (0.6) we reduce calculations of the integral to quasiclassical calculations for $t\to\infty$:

$$Z(0) = \lim_{t\to\infty} Z(t) = \sum_{k,r}\frac{t^r}{k!r!}\int_M e^{i(H+th)}\tilde{\Omega}^r\Omega^m\,, \tag{0.7}$$

where $\tilde{\Omega} = d\omega$, $h = \iota_{\mathbf{K}}\omega$. Now calculate partition function at $t\to\infty$. $dh = d(\iota_{\mathbf{K}}\omega) = -\iota_{\mathbf{K}}\tilde{\Omega}$. Hence at zero locus of $\mathbf{K}$, i.e. $dh = 0$ we have

$$\mathrm{Hess}\,H\big|_{x_i} = \frac{\partial^2 H}{\partial x^m \partial x^n}\big|_{x_i} = \tilde{\Omega}_{mn}\big|_{x_i}\,. \tag{0.8}$$

Hence using the fact that for symmetric bilinear form $A(\mathbf{x},\mathbf{x})$ in k-dimensional Euclidean space $\mathbf{R}^k$

$$\int_{\mathbf{R}^k} e^{itA(\mathbf{x},\mathbf{x})}d^kx = \int_{\mathbf{R}^k} e^{itA_{ij}x^ix^j}d^kx = \frac{e^{\frac{i\pi k}{4}}\sqrt{\pi^k}}{t^{\frac{k}{2}}\sqrt{\det A}}\,,$$

we obtain that at the quasiclassical limit for partition function $Z(t)$ in (0.7) is equal to

$$\lim_{t\to\infty} Z(t) = \sum_{r=0}^{n}\frac{t^r}{(n-r)!r!}\int_M e^{i(H+th)}\tilde{\Omega}^r\Omega^{n-r} =$$

$$\lim_{t\to\infty}\sum_{r=0}^{n}\sum_{x_i}\frac{t^r}{(n-r)!r!}\frac{e^{iH}\sqrt{\det\tilde{\Omega}_{ik}}}{t^n\sqrt{\det\mathrm{Hess}\,H}}\Big|_{x_i} = \sum_{x_i}\frac{e^{iH}\sqrt{\det\tilde{\Omega}_{ik}}}{\sqrt{\det\mathrm{Hess}\,H}}\Big|_{x_i}$$

Now choose ω such that $\tilde{\Omega} = d\omega$ is non-degenerate at locus of K. We have $dh = \iota_{\mathbf{K}}\tilde{\Omega}$. Hence at locus of $\mathbf{K}$

$$\mathrm{Hess}\, H = \frac{\partial^2 H(x)}{\partial x^m x^n} = \tilde{\Omega}_{mr}\frac{\partial K^r}{\partial x^n}\,,$$

and we have finally that

$$\lim_{t\to\infty} Z(t) = \sum_{x_i} \frac{e^{iH}\sqrt{\det\tilde{\Omega}_{ik}}}{\sqrt{\det \mathrm{Hess}\, H}}\Big|_{x_i} \sum_{x_i} \frac{e^{iH}}{\sqrt{\det\frac{\partial K}{\partial x}}}\Big|_{x_i}$$

Thus due to relation (0.6) leads to (0.3).

Remark 1 The form $\tilde{\Omega} = d\omega$ and new Hamiltonian $h = \iota_{\mathbf{K}}\omega$ define the same Hamiltonian vector field $\mathbf{K}$ as a pair (Ω, H). On the other hand the pair $(\tilde{\Omega}, \omega)$ is more suitable for calculation of quasiclassical approximation. The $U(1)$-vector field $\mathbf{K}$ is fundamental object of DH-localisation formula, not the pair which produces this field (see in detail §2).

Remark 2

One of the way to produce $\mathbf{K}$-invariant form ω is the following: One can take ω-covector $\mathbf{K}$ with respect to $U(1)$-invariant metric: $\omega = \omega_i dx^i$, $w_i = g_{ik}K^k$ and g_{ik} is $U(1)$-invariant Riemannian metric (average over group $U(1)$). It is crucial for calculation that $\tilde{\Omega} = d\omega$ is non-degenerate at zero locus of $\mathbf{K}$. Is it an additional condition, or it follows from the fact that vector field $\mathbf{K}$ generates $U(1)$-action (and M is even-dimensional manifold)? On one hand I cannot prove this completely, on the other hand natural counterexamples deal with non-compact vector field.

§1 DH-formula and supersymmetric mechanics. Nersessian's approach.

The considerations of this paragraph are based on the work [2]

The calculations above can be put in supersymmetric framework. Differential form on M can be considered as a function on ΠTM—tangent bundle to M with reversed parity of fibers $w_i(x)dx^i \to w_i(x)\xi^i, \ldots$. Integral of form over M is the integral of a function over supermanifold ΠTM with invariant volume form $dx^1 \ldots dx^{2n} d\xi^1 \ldots d\xi^{2n}$.

In the very nice paper [1] Armen Nersessian suggested the supersymmetric framework of the calculations above. I will try to explain it here. Recall that for an arbitrary Poisson manifold M (manifold with Poisson bracket $\{\ ,\ \}$) one can consider odd Koszul bracket $[\ ,\ ]$ on ΠTM such that for arbitrary functions f, g on M we have that

$$[f, g] = 0\,, [f, dg] = \{f, g\}\ [df, dg] = d\{f, g\}\,. \tag{1.1}$$

In local coordinates $[x^i, x^k] = 0$, $[x^i, \xi^k] = \Omega^{ik}$, $[\xi^i, \xi^k] = \xi^r\partial_r\Omega^{ik}$.

If Poisson structure is symplectic one then

$$[\Omega, F] = dF\,, \qquad (\Omega = \Omega_{ik}\xi^i\xi^k) \tag{1.2}$$

If H is an arbitrary Hamiltonian on M and $\mathbf{K} = D_H$ Hamiltonian vector field then

$$[H, F] = \iota_{\mathbf{K}} F \tag{1.3}$$

We see that

$$(d+\iota_k)F = [\Omega + H, F]$$

and

$$\mathcal{L}_{\mathbf{K}}F = (d+\iota_{\mathbf{K}})^2 = [H+\Omega, [H+\Omega, F]] = [[H,\Omega], F]\,.$$

Thus we come to core of Dustermaat-Heckman formalism:

Form F is invariant with respect to odd vector field $d_{\mathbf{K}} = d + \iota_{\mathbf{K}}$ if it is integral of motion of 'Hamiltonian' $H+\Omega$, form F is invariant with respect to Hamiltonian vector field $\mathbf{K} = D_H$ if it is integral of motion of 'Hamiltonian' $G = [H, F]$.

The partition function (0.5) can be rewritten as

$$Z(t) = \int e^{i(H-\Omega-t[H+\Omega,\tilde{G}])}\,.$$

Remark 3 Hamiltonians $\{H+\Omega, H-\Omega, \Omega\}$ form superalgebra.

§2**Schwarz-Zaboronsky supersymmetric formalism**

In this paragraph we will speak about approach developed in the paper [2], where supergeometry is powerfully used for formulating localisation formula in a more general case.

It will always be assumed that M is compact manifold and $\mathbf{K}$ is compact vector field on it, i.e. vector field which generates $U(1)$ action. We denote by

$$Q_{\mathbf{K}} = d + \iota_{\mathbf{K}}\,, \quad \text{in "supernotations"}\ Q_{\mathbf{K}} = \xi^i \frac{\partial}{\partial x^i} + K^i(x)\frac{\partial}{\partial \xi^i}\,,$$

where $x^i, \xi^i = dx^i$ are local coordinates on ΠTM.

Odd vector field $Q_{\mathbf{K}}$ is a "square root" of a Lie derivative $\mathcal{L}_K = \iota_{\mathbf{K}} \circ d + d \circ \iota_{\mathbf{K}}$:

$$\mathcal{L}_{\mathbf{K}} = Q_{\mathbf{K}}^2 = \left(\xi^i \frac{\partial}{\partial x^i} + K^i(x)\frac{\partial}{\partial \xi^i}\right)^2 = K^i(x)\frac{\partial}{\partial x^i} + \xi^r \frac{\partial K^i}{\partial \xi^r}\frac{\partial}{\partial \xi^i}\,, \tag{1}$$

or in classical notations

$$\mathcal{L}_{\mathbf{K}} = Q_{\mathbf{K}}^2 = (d+\iota_k)^2 = d\circ\iota_{\mathbf{K}} + \iota_{\mathbf{K}}\circ d\,.$$

We formulate the following version of DH localisation theorem:

Theorem *Let $H = H(x, dx)$ be a $Q_{\mathbf{K}}$-invariant form on M, i.e.*

$$dH + \iota_{\mathbf{K}} H = 0\,. \tag{2}$$

Then the integral $\int_M H(x,dx)$ is localised at locus of K. This means follows: let U_K be an arbitrary $U(1)$-invariant tubular neighborhood of locus of K and let $G_U = G_U(x, dx)$ be a*

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* *the condition to be $U(1)$-invariant may be is not necessary. We will use it for constructing $U(1)$-invariant partition of unity. This condition is absent in the paper [1].*

$Q_{\mathbf{K}}$-invariant form such that it is equal to 1 at the locus of vector field $\mathbf{K}$ and it vanishes out of neighborhood $U_{\mathbf{K}}$:

$$Q_{\mathbf{K}} G_U = 0, \text{(i.e.} \, dG_U + \iota_{\mathbf{K}} G_U = 0), \quad G_U\big|_{locus\ of\ \mathbf{K}} = 1, \quad G_U\big|_{M \setminus U_K} = 0 \,. \tag{3}$$

(Bump-form of zero locus of $\mathbf{K}$.) (We will prove the existence of such a bump-form)

Then

$$\int_M H = \int_M H G_U \,. \tag{4}$$

Example Let M be a symplectic manifold, i.e. non-degenerate closed two-form Ω is defined on M (M is even-dimensional). Let $h = h(x)$ be a Hamiltonian such that its Hamiltonian vector field D_h (D_h: $\iota_{D_h} \Omega = -dh$) is compact, i.e. it defines $U(1)$ action. Consider the form

$$H(x, dx) = \exp i \left(\Omega + h\right) \,. \tag{5}$$

This form is $Q_{\mathbf{K}}$-invariant. Indeed since K is Hamiltonian vector field D_h hence

$$\iota_{\mathbf{K}} \Omega + dh = 0 \text{.i.e.} \, Q_{\mathbf{K}}(h + \Omega) = 0 \Rightarrow Q_{\mathbf{K}} H = 0 \,.$$

Then

$$\int H(x, dx) = \int \exp i \left(\Omega + h\right) = \frac{i^n}{n!} \int \exp ih \underbrace{\Omega \wedge \ldots \wedge \Omega}_{n \text{ times}}$$

is localised.

Remark 4 Note that this example is a basic example in classical background. Compact vector field $\mathbf{K}$ appears naturally in this example as Hamiltonian vector field of Hamiltonin h. In Schwarz-Zaboronsky approach the vector field $\mathbf{K}$ appears independently without symplectic structure and Hamiltonian. In this approach the localisation formula is stated for a function $H(x, dx)$ on ΠTM (sum of differential forms of different orders). The classical condition that sum of differential forms is invariant with respect to equivariant differential $d_{\mathbf{K}} = d + \iota_{\mathbf{K}}$ becomes the condition that "function" [2]

$H(x, dx)$ is invariant with respect to odd vector field $Q_{\mathbf{K}}$ which is the square root of Lie derivative along the vector field $\mathbf{K} : Q_K^2 = \mathcal{L}_{\mathbf{K}}$.

Remark 5 'Super-language' becomes essentially important for constructing of partition of unity for forms.

Proof of Theorem First we prove the existence of a form $G_U = G_U(x, dx)$ which obeys the condition (3), then we will show that an arbitrary $Q_{\mathbf{K}}$-invariant "function" (form) which obeys conditions (3) yields the localisation formula (4).

Using partition of unity arguments consider a function $F = F(x)$ such that

$$F(x)\big|_{\text{locus of } \mathbf{K}} = 0, \quad F(x)\big|_{M \setminus U_K} = 1 \,. \tag{6}$$

[2] $H(x, dx)$ is non-homogeneous differential form on M. It is a function on tangent bundle ΠTM with reversed parity of fibers.

(We may consider partition of unity which is subordinate to covering $V_1 \cup V_2$, where $V_1 = U_{\mathbf{K}}$ and $V_2 = M\backslash$locus of K.

We may assume that $F(x)$ is $\mathbf{K}$-invariant function. (Here we use the $U(1)$-invariance of neighbourhood of locus (see the footnote.)).

It is useful to consider the differential 1-form

$$\omega_{\mathbf{K}}\colon \omega_{\mathbf{K}}(\mathbf{x})\langle \mathbf{K}, , \mathbf{x}\rangle\,, \omega_i = g_{im} K^m dx^i\,, \tag{7}$$

where $\langle \mathbf{K}, , \mathbf{x}\rangle$ is $U(1)$-invariant Riemannian metric on M. Now we are ready to define form G_U which obeys the condition (3):

$$G_U(x, dx) = 1 - Q_{\mathbf{K}}\left(\frac{\omega_{\mathbf{K}}(x, dx)}{Q_{\mathbf{K}}\omega_{\mathbf{K}}} F(x)\right) \tag{8}$$

Straightforward calculations show that this function obeys conditions (3). Indeed $F(x) = 0$ if x belongs to locus of K (and in a vicinity of the locus), hence the right hand side of equation (8) is well-defined on the locus of $\mathbf{K}$, where the form $\omega_{\mathbf{K}}$ is not defined. Using the fact that $Q_{\mathbf{K}}\left(\frac{\omega_{\mathbf{K}}(x,dx)}{Q_{\mathbf{K}}\omega_{\mathbf{K}}}\right) = 1$ (if $\mathbf{K}(x) \neq 0$) we immediately come to the condition (3).

Let $\tilde{G}_U = \tilde{G}_U(x, dx)$ be an arbitrary $Q_{\mathbf{K}}$-invariant form which obeys the condition (3). Then consider the difference $L(x, dx) = \tilde{G}_U - G_U$. The form $L(x, dx)$ is $Q_{\mathbf{K}}$-invariant and it is equal to 0 at the locus of K, Hence

$$L(x, dx) = Q_{\mathbf{K}}\left(\frac{\omega_{\mathbf{K}}(x, dx)}{Q_{\mathbf{K}}\omega_{\mathbf{K}}} L(x, dx)\right)\,. \tag{9}$$

Thus we see that $Q_{\mathbf{K}}$-invariant form $G_U(x, dx)$ in (8) which obeys the condition (3) as well as an arbitrary $Q_{\mathbf{K}}$-invariatn form $\tilde{G}_U(x, dx)$ which obeys the condtion (3) obey the condition that

$$\begin{array}{c} G_U(x, dx) = 1 + Q_{\mathbf{K}}(...) \\ \tilde{G}_U(x, dx) = 1 + Q_{\mathbf{K}}(...) \end{array}$$

This immediately implies the relation (4):

$$\int_M H(x, dx) G_U(x, dx) = \int_M H(x, dx)(1 + Q_{\mathbf{K}}(\ldots)) = \int_M H(x, dx)$$

since $\int_M Q_{\mathbf{K}}(\ldots) = 0^{**}$ ■

Concrete calculations

Now based on the Theorem we present concrete calculations. which are very similar to calculations in paragraph 0.

** since $Q_K = d + \iota_K$, and $\iota_K \omega$ 'does not contain' top form. This follows also from the vanishing of divergence of odd vector field $Q_{\mathbf{K}}$ with respect to canonical volume form in ΠTM

Let $H = H(x, dx)$ be $Q_{\mathbf{K}}$ invariant form and locus (zero locus) of $U(1)$-invariant vector field $\mathbf{K}$ is a set $\{x_i\}$ of isolated points.

Using bump-form G_U, the form which vanishes out vicinities of points $\{x_i\}$ (see the considerations above) we calculate $\int_M H(x, dx)$.

Lemma For an arbitrary $Q_{\mathbf{K}}$-invariant form $H(x, dx)$ the integral

$$Z(t) = \int H(x, dx) e^{itQ_{\mathbf{K}}(\omega_{\mathbf{K}})}\,,$$

where $\omega_{\mathbf{K}}$ is $U(1)$-invariant form (7) does not depend on t.

Proof:

$$\frac{dZ(t)}{dt} = i \int_M H(x, dx) Q_{\mathbf{K}} \omega_{\mathbf{K}} e^{itQ_{\mathbf{K}}(\omega_{\mathbf{K}})} = i \int_M Q_{\mathbf{K}} \left(H(x, dx) e^{itQ_{\mathbf{K}}(\omega_{\mathbf{K}})} \right) = 0\,.$$

Now using lemma and bump-form which localises integrand in vicinity of points $\{x_i\}$ we come to

$$\int_M H(x, dx) = \int_M H(x, dx) G_U(x, dx) = \left(\int_M H(x, dx) G_U(x, dx) e^{itQ_{\mathbf{K}}(\omega_{\mathbf{K}})} \right) \big|_{t=0}$$

$$= \left(\int_M H(x, dx) G_U(x, dx) e^{itQ_{\mathbf{K}}(\omega_{\mathbf{K}})} \right) \big|_{t\to\infty}$$

Using method of stationary phase and assuming that $d\omega$ is non-degenerate at locus of $\mathbf{K}^*$ we calculate the last integral (see [4]) and come to the answer

$$\int_M H(x, dx) == \left(\int_M H(x, dx) G_U(x, dx) e^{itQ_{\mathbf{K}}(\omega_{\mathbf{K}})} \right) \big|_{t\to\infty} = \sum_{x_i} \frac{i^n}{n!} \frac{H(x, dx)\big|_{x_i}}{\sqrt{\frac{\partial K}{\partial x}\big|_{x_i}}}$$

If $H(x, dx)\big|_{x_i} = H_0(x_i)$, where $H(x, dx) = H_0(x) + H_1(x, dx) + \ldots$ is a sum of differential forms.

References

[1] A. Nersessian *Antibrackets and localisation of (path) integrals* arXix: hep-th/9305181, (published in JETP)

[2] Albert Schwarz and Oleg Zaboronsky. *Supersymmetry and localisation.* arXiv: hep-th/951112v1, (published in CMP)

[3] *On the Duistermaat-Heckman localisation formula and Integrable systems* arXiv: hep-th/9402041v1

[4] homepage: maths.manchester.ac.uk/khudian/Etudes/Geometry/Dustermaat-Heckman localisation formula. *Etude based on the fragment of the lecture of A.Belavin in Bialoveza, summer 2012.*

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* See the remark 2