Cube and tetrahedron are not equipartial.

Theorem 1 Two polygons of equal area are equipartial.

This means that if polygons Π_1 and Π_2 have the same area then one can cut the polygon Π_1 on polygons $\pi_1, \dots, \pi_k$ and polygon Π_2 on polygons $\pi'_1, \dots, \pi'_k$ such that polygons π_k are equal to polygons π'_k: $\pi_1 = \pi'_1$, $\pi_2 = \pi'_2$, $\pi_3 = \pi'_3, \dots, \pi_k = \pi'_k$.

The proof is simple. I give two hints to prove it.

Hint 1. This was proved by amateur mathematician in XIX century. This means that you can prove it! (Ja v svojo vremia sdelal eto s udovoljstvijem!)

Hint 2. The proof immediately follows from the lemma.

Lemma Let S_1 be a triangle (with acute angles), and S_2 be an rectangtle such that they have the same area and one of the sides of triangle S_1 coincides with one of the sides of the rectangle S_2. Then the triangle S_1 is equipartial with the rectangle S_2.

Proof: Let S_1 be a $\triangle ABC$ with $a = BC$ and S_2 be rectangle with a side a. Consider the segment MN joining midpoints M, N of the sides AB and AC, and the altitude (height) AP of the triangle AMN. Then cut triangle ABC on $\triangle AMP$, $\triangle ANP$ and trapezoid $BMNC$. Puting triangles ABC, AMP to the trapezoid we come to the rectangle.

Now it is easy to prove the Theorem. To see how the lemma helps consider

Example. Show that rectangle Π_1 with sides $\{1, 2\}$ and square Π_2 with sides $\{\sqrt{2}, \sqrt{2}\}$ are equipartial.

Solution: It follows from lemma that rectangle Π_1 with sides $\{1, 2\}$ and triangle with sides $\{2, 2, 2\sqrt{2}\}$ are equipartial. Again applying lemma we see that triangle with sides $\{2, 2, 2\sqrt{2}\}$ and rectangle Π_3 with sides $\{2\sqrt{2}, \frac{\sqrt{2}}{2}\}$ are equiaprtial. On the other hand rectangle Π_3 with sides $\{2\sqrt{2}, \frac{\sqrt{2}}{2}\}$ and square Π_2 with sides $\{\sqrt{2}, \sqrt{2}\}$ are equipartial. Hence rectangle Π_1 and square Π_2 are equiaprtial.

Now the most interesting part:

Theorem 2 The cub and tetrahedron of the same volume are not equipartial.

It is one of the Hilbert's problem.

The meaning of this theorems is following: we know that area of the triangle is equal to $S = \frac{ah}{2}$, where h–is the length of the altitude on the side a; and the volume of tetrahedron is equal to $\frac{SH}{3}$, where S is the area of the base and h is the length of the altitude on the base. The Theorem 2 means that one *cannot escape the Analysis* (consider integration) to define the volume of tetrahedron*.

Few weeks ago I heard about wonderful proof of the second Theorem. (Davidik rasskazal mne eto dokazateljstvo, kogda ja vstretilsja s nim na Ukrajine. On priivjoz eto iz Moskvy) Here it is:

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* The Theorem 1 claims that one comes to the formula for an area of triangle just by cutting rectangle and *without using Analysis*, i.e. without integration

Consider cube with edge 1 and regular tetrahedron of the same volume. Let θ be an angle between sides of the tetrahedron. One can see that $\frac{\theta}{\pi}$ is irrational number. (I think this follows easy from the fact that $\cos\theta = 1/3$).

For every polyhedron C consider the function

$$P_C = \sum_i |l_i| F(\varphi_i), \tag{1}$$

where $\{l_i\}$ are edges of the polyhedron, φ_i is the angle between sides adjusted to the edge l_i and $F(\varphi)$–a real valued function (v etoj funktsiji i vsja solj!). The summation goes over all edges l_i.

Now the most interesting part: Consider an additive function F on $\mathbf{R}$: $F(a+b) = F(a)+F(b)$, i.e. the linear function on the real numbers, considered as a vector space over rational numbers, such that

$$F(\pi/2) = 0\,, \tag{2}$$

and

$$F(\theta) = 1. \tag{3}$$

One can see that condition (2) implies that function (1) is not changed under cuttings.

This function exists because $\frac{\theta}{\pi}$ is irrational number, but this function is not linear in common sense, i.e. it is not *continuous* function.! To construct this function we need Hamel basis **. Now the proof is in one line:

The function $P_C = \sum_i |l_i| F(\varphi_i)$ defined by relations (1),(2) and (3) is equal to 0 if C is the cube of volume 1 and it is equal to $z = 6l$ if C is regular tetrahedron, where l is a length of the teathredon. On the other hand the function P_C does not change under cutting of polyhedron because the function F is additive function of the angles, and the condition $F(\pi) = 0$ is obeyed. Contradiction. ■

I enjoyed so much this proof, but something is worrying: we use Choice Axiom for constructing additive not continuous function F on all real numbers. Do we really need it?

I think one can escape the using of choice axiom.

Indeed suppose one can cut cube on polyhedra $\gamma_1, \dots, \gamma_k$ such that after putting with each other we come to tetrahedron. Consider the finite set of angles $\{\varphi_1, \dots, \varphi_N\}$ which arise during cuttings.

Let V be the linear space spanned by the numbers $\{\varphi_1, \dots, \varphi_N\}$ with rational coefficients:

$$V = \{a_1\varphi_1 + \ldots + a_N\varphi_N, \ \text{where } a_1, \dots, a_n \in \mathbf{Q}\}$$

** The space $\mathbf{R}$ of real numbers is a vector space over rational numbers. The basis in this space is the set $\{e_\alpha\}$ of numbers such that for an arbitrary real number $\mathbf{b}$, $\mathbf{b} = \sum \gamma_\alpha e_\alpha$ where all $\{\gamma_\alpha\}$ are equal to zero except the finite set. The set $\{\gamma_\alpha\}$ is defined uniquely. The problem is that this vector space is "worse" than infinite-dimensional— its dimension is uncountable. To find a basis $\{\mathbf{e}_\alpha\}$ one needs to use transfinite induction, i.e. essentially use of Choice Axiom. A basis $\{e_\alpha\}$ is called *Hamel basis*

Let F be a linear function on V which obeys the condition (2). One does not need Choice Axiom to construct this function (in spite of the fact that a function F is not defined uniquely), since V is finite-dimensional vector space. It suffices to consider this function to come to contradiction.

Krassivo nepravda li?

Remark (28.12.2013). The proof of this Theorem is founded on the function $I = \sum |l_i| F(\varphi_i)$ where $F(\varphi_i)$ is a function which is linear on a finite sequence of angles φ_α, where φ_α are all the angles which arise during cuttings of cube.

Then the restriction $F(\pi) = 0$ implies that I is invariant of cuttings...

I would like to mention that if we take the genuine linear function $F(\varphi) = \varphi$ then these considerations will fail (cutting around faces will change the invarance) but we will come to...

$$I(M) = \sum |l_i| F(\varphi_i) = \int_M H$$

where H is mean curvature of closed surface M. It is interesting the interplay of this formula with tube formula

$$V_h = V_0 + S_m h + h^2 \int_M H + \frac{4}{3}\pi h^3$$

where $M = \partial D$ is a boundary of convex domain, and $V_h(D)$ is a volume of all the body which contains all the points of D + points which are on the distance from M less or equal to h, and h is enough small.