On one subset in $SO(3)$ and Euler Theorem

This text was written around 2006. Korchagina had a beautiful talk with very facinating title "On the second generation of proof of Theorem on finite simple groups". Sasha Borovik did some interesting remarks about status of this Theorem and about relation between $\mathbf{R}P^3$ and $SO(3)$.

around 2006 — *The group $S0(3)$ has fantom memories of lost operations (...initiated by remark of Sasha Borovik)* $\blacksquare$

It is well-well-known that $SO(3) \approx \mathbf{R}P^3$(as manifold) , but in fact one can say much more: $S0(3)$ 'knows' about structure of projective space. In particular the subspace $\mathbf{R}P^2 \subset \mathbf{R}P^3$ can be canonically embedded in $SO(3)$. It is the following subset (not subgroup!) in $SO(3)$:

$$L = \{A\colon\ A \in SO(3),\, \det(1 + A) = 0\}\,. \tag{1}$$

Geometrically it means following: Operators $A \in L$ are orthogonal transformations which are rotations around axis on the angle π. It will be $\mathbf{R}P^2$[1] . This is not hard to see. The subset Lh naturally appears also in another algebraic proof of Euler Euler Theorem.

Recall first maybe the most beautiful proof of Euler Theorem (Coxeter Proof). Let '$A \in SO(3)$, $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ be an orthogonal basis and $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ be a new orthogonal basis, $\mathbf{e}' = A\mathbf{e}, \mathbf{f}' = A\mathbf{e}, \mathbf{g}' = A\mathbf{g}$. Consider a reflection O_1 which transforms $\mathbf{e}$ to $\mathbf{e}'$; the invariant plane of this reflection is spanned by vectors $\mathbf{e} + \mathbf{e}'$ and $\mathbf{e} \times \mathbf{e}'$. Then consider the reflection O_2 which transforms $\mathbf{f}$ to $\mathbf{f}'$. The vector $\mathbf{e}'$ belongs to invariant plane of this reflection. Under composition of these two reflections f transforms to f' too. Indeed in other case we consider third reflection with respect to the plane $\mathbf{e}', \mathbf{f}'$ but composition of three reflections has determinant -1.) Hence $A = O_2 O_1$. The intersection of invariant planes of these reflection is an axis of rotation. $\blacksquare$

Now algebraic proof. Let $A \in SO(3)$. Note first that if there exists eigenvector $\mathbf{n}$ with eigenvalue 1 then restriction of transformation A on the plane orthogonal to the vector $\mathbf{n}$ is nothing but orthogonal rotation of the plane α. (Indeed if α is a plane which is orthogonal to $\mathbf{n}$, then it is obvious that A maps α on α, it is orthogonal operator on α and $\det A|_\alpha = 1$.). Hence A is rotation around axis $\mathbf{n}$. It remains to prove that such an eigenvector $\mathbf{n}$ exists. Consider characteristic polynomial $P(z) = \det(z - A)$. It is cubic polynomial, and it has at least one real root. Let λ be its real root and $\mathbf{n}$ be corresponding eigenvector. Operator A preserves scalar product. This implies that $\lambda = \pm 1$. if $\lambda = 1$,

---

[1] Any rotation =axis+angle of rotation=point of the ball of the radius π. Rotation on angle $\neq \pi$ is an interior of this ball. The boundary of this ball with fatorised antipodal points $= L$

then everything is already proved (see above). If $\lambda =: -1$, then $\det(1+L) = 0$, i.e. $A \in L$ [2]. Let $\mathbf{n}$ be an eigenvector corresponding to the eigenvalue $\lambda = -1$: $A\mathbf{n} = -\mathbf{n}$. Consider a plane α, which is orthogonal to the eigenvector $\mathbf{n}$. Restriction of A on the plane α is orthogonal transformation with determinant -1, i.e. the restriction of A on the plane α is a reflection of the plane with respect of a line $l \in \alpha$ (A on α has eigenvectors $\mathbf{f}, \mathbf{g}$ with eigenvalues $1, -1$ respectively, and l is directed along vector $\mathbf{f}$.) We see that in this special case A is a rotation around axis λ on the angle π. Subset L is in one-one correspondence with axis= lines which go over origin, i.e. $L = \mathbf{R}P^2$. ■

[2] Of course in this case also one can prove the existence of eigenvector with eigenvalue 1, but it is much easier to do it straightforwardly