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Hidden hyperbolicity

In this étude we consider one geometrical problem, which has two manifestations. It seems to be the standard Euclidean problem, but it possesses the hidden hyperbolicity. We first explain how the example is arised, then consider its solution in terms of Euclidean and Hyperbolic Geometry

Source

Consider a realisation of hyperbolic (Lobachevsky) plane as upper half-plane of Euclidean plane. Let C be a (usual Euclidean) circle in this half-plane. It is also the hyperbolic circle/ Let O be a centre of C considered as the Eucldean circle, and let A be a centre of C considered as the hyperbolic circle. The fact that the point A is the hyp[erbolic centre means that if two curves γ, γ' are arbitrary hyperbolic geodesics passing through the point A, and $L_\gamma, L'_{\gamma'}$ are points of the intersections of these geodesics with circle C, $L_\gamma = \gamma \times C$, $L_{\gamma'} = \gamma' \times C$ then the (hyperbolic) lengths of these geodesics coincide. In particular due to the lemma this implies that all these geodesics intersect the circle C under the angle $\frac{\pi}{2}$.

Now look at this picture from the point of view of Euclidean geometry. All gedesics are half-circles with centres on the absolute, the line $x = 0$, (except the geodesicis which are the vertical lines) Angles are the same. If geodesic γ is represented by the half-circle with the centre at the point K, then the segment KL_γ, radius of this half-circle has to be tangent to the circle C in the case if this geodesic intersects the circle C under the angle $\frac{\pi}{2}$. Thus we come to conclusion that for the arbitrary point K on the absolute the (Euclidean) length of the tangent from the point K to the circle C is equal to the (Euclidean) length of the segment KA.

Based on the considerations above we will formulate the following problem of Eucldean geometry.

Let C be a circle in the Euclidean plane and let A, be an arbitrary point on this plane.

Consider the locus $M_{C,A}$ of the points K such that the length of the tangent from K to the circle C is equal to the length of the segment KA:

$$M_{C,A} = \{K\colon \quad KA = \textit{lenght of the tangent from } K \textit{ to the circle.}\}\,. \tag{1}$$

Find the locus $M_{C,A}$.

This problem looks as standard geometrical question in Euclidean gepmetry. Temporary forgetting where this problem comes from, we will discuss first solution of this problem just in terms of Eucldean geometry

Solution

Let A' be a point which is inverse to the point A with respect to the circle C: Points O (centre of the circle), A and A" belong to the same ray r_{OA} and

$$|OA| \cdot |OA'| = 1\,. \tag{2}$$

(We suppose that the circle C has unit length.)

Consider the set of circles passing through the points A and A'. One can see that every such a circle,

$$\text{intersects the circle } C \text{ under the right angle} \tag{3}$$

This means that for centre K of such circle, the tangent to the circle C is the radius of this circle. We see that the locus $M_{C,A}$ = the locus of the centres of circles passing through the points A and A' = the locus of the points which are on the same (Euclidean) distance from the points A and A'. This is the line l which is ortogonal to the AA' and passes through th emiddle point P of the segment AA' ($P \in AA', |AP| = |PA'|$.

So we come to

$$M_{C,A} = \{l: \quad d(l, A) = d(l, A')\}\,. \tag{4}$$

It remains to prove just relation (3). It can be checked straightforwardly. The following beautiful prove implies inversion.

Let L be an arbitrary circle passing through the points A, A'. Let this circle intersects the circle C at the point M. Triangle AMA' remains intact under the inversion with respect to the circle C: $A \leftrightarrow A', M \leftrightarrow M$ (see equation (1)). Hence the inversion transforms the circle L to the same circle. If φ is angle of intersection of L with C, then $\pi - \varphi$ is the angle of intersection of inversed circle with C;ince inversion transforms the direction of arcs. We have $\varphi = \pi - \varphi$, i.e. $\varphi = \frac{\pi}{2}$.

Meaning in hyperbolic geometry

The considerations of the first paragraph show that the fact that $l = M_{C,A}$ has explanations in hyperbolicity.

Indeed the line l divides the plane on two half-planes. Consider the half-plane the circle C belogs to, as a model of hyperbolic (Lobachevsky) plane *. The circle C will be the circle in the hyperbolic plane also.

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* Recall shrotly what is it. One can consider Cartesian coordinates (x, y) such that the line l is $y = 0$ the half-plane is $y \geq 0$. Then hyperbolic plane H can be defined as as this half-plane with Riemannian metric $G = \frac{dx^2+dy^2}{y^2}$. The geodesics of this metric (lines of hyperbolic plane) are vertical lines $x = a$ and upper half-circles with centre on the absolute l: $\begin{cases} (x-a)^2 + y^2 = R^2 \\ y > 0 \end{cases}$. The distance between two points $A_1 = (x_1, y_1)$

One of the points A or A' is in the circle C. WLOG suppose that this is a point A.

One can see traightfoewardly that the point A is the centre of the hyperbolic circle C [2]. This immediately implies that the points of l belong to the locus $M_{C,A}$. Indeed due to the lemma all the geodesics starting at the point A, the centre of the circle, interesect the circle C under the right angle, i.e. the points of l belong to this locus.

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and $A_2 = (x_2, y_2)$, the length of the geodesic passing via points A_1, A_2 can be defined alternatively by cross-ratio of the points

$$d(A_1, A_2) = |\log(A_1, A_2, A_0, A_\infty))| = \left|\log\left(\frac{z_1 - z_0}{z_1 - z_\infty} : \frac{z_2 - z_0}{z_1 - z_\infty}\right)\right| ,$$

where points A_0, A_∞ are points of intersection of the half-circle with absolute, $z_1 = x_1 + iy_1$,$z_2 = x_2 + iy_1$. E.g. for two points $A_1 = (0, a_1)$, $A_2 = (o, a_2)$ on the vertical line (this is geodesic)

$$d(A_1, A_2) = |\log|(A_1, A_2, 0, \infty))| = \left|\log\left(\frac{ia_1 - 0}{ia_1 - \infty} : \frac{ia_2 - 0}{ia_1 - \infty}\right)\right| = \left|\log\left(\frac{a_1}{a_2}\right)\right| .$$

[2] It is convenient to consider coordinates (x, y) such that the line l is defined by $x = 0$, and the vertical ray AP is $y = 0$. Let point a be on the distance a from the Euclidean centre of the circle C. Then the Euclidean centre of the circle has coordinates $O = \left(0, \frac{1}{2}\left(a + \frac{1}{a}\right)\right)$, and respectively

$$P = (0, 0) , \quad A = \left(0, \frac{1}{2}\left(a + \frac{1}{a}\right) - a\right) = \left(0, \frac{1}{2}\left(\frac{1}{a} - a\right)\right) .$$

Recall that P is the point of intersection of the vertical ray passing throught the point A with absolute l. Let $N_{1.2}$ be points of the intersection of the ray AP, then

$$N_{1,2} = \left(0, \frac{1}{2}\left(a + \frac{1}{a}\right) \pm 1\right) = \left(0, \frac{1}{2}\left(\sqrt{\frac{1}{a}} \pm \sqrt{a}\right)^2\right)$$

We see that $|PA|$ is geometrical mean of PN_1, PN_2:

$$|PN_1| \cdot |PN_2| = \frac{1}{2}\left(\sqrt{\frac{1}{a}} + \sqrt{a}\right)^2 \frac{1}{2}\left(\sqrt{\frac{1}{a}} - \sqrt{a}\right)^2 = \frac{1}{4}\left(\frac{1}{a} - a\right)^2 = |PA|^2 ,$$

i.e. the hyperbolical lengths $|AN_1|$ and $|AN_2|$ coincide.