**Symmetries of Clairaut equation**

**or**

**how looks infinitesimal Legendre transformations**

*Recall that ordinary first order differential equation* $f(x,y',y)=0$ *has the following geometrical interpretation: The difequation defines surface* $M_f$ *of codimension* 1 *in the contact space* $J^1(\mathbf{R})$ *with coordinates* $(x,p,y)$ $(p=y')$*,*

$$M_f=\{(x,p,y)\colon f(x,p,y)=0\} \tag{0.1}$$

*and the (generalised) solution of the equation is a curve* $l=(x(t),p(t),y(t))$ *belonging to this surface such that the contact form*

$$\omega = pdx - dy \tag{0.2}$$

*vanishes on the curve* [1)]

*The Legendre transformation*

$$\tilde x = p, \tilde p = -x, \tilde y = y - px\,, \tag{0.3}$$

*of contact space* $J^1(\mathbf{R})$ *preserves the contact form.*

*We study here wow lookz infinitesimal transformation which generates the Legendre transformation. This maybe applied to Clairaut equation.*

Our aim to find the one-parametric famiily of contact trasnformations which include the Legendre transformation (0.3)

Recall that infinitesimal transformations—vector fields on $J^1(\mathbf{R})$ which preserves the distribution of the planes which vanish the contact form (0.2) are in one-one correspondence with Hamiltonians on $J^1(\mathbf{R})$:

$$C(J^1(\mathbf{R}))\ni H(x,p,y)\leftrightarrow \mathbf{X}\colon\quad \mathcal{L}_{\mathbf{X}}(pdx-dy)=\lambda(x,p,y)(pdx-dy)\,. \tag{1.1}$$

The correspondence is the following:

$$H(x,p,y)\mapsto \mathbf{X}_H=\frac{\partial H}{\partial p}\frac{\partial}{\partial x}-\frac{\partial H}{\partial x}\frac{\partial}{\partial p}+\left(p\frac{\partial H}{\partial p}-H\right)\frac{\partial}{\partial y}\,,\quad \mathbf{X}\mapsto H=\omega\rfloor\mathbf{X}\,,(\lambda=-H_y)\,. \tag{1.2}$$

Consider on $J^1(\mathbf{R})$ the Hamiltonian $H=\frac{p^2}{2}+\frac{x^2}{2}$ of harmonic oscillator. This Hamiltonian defines due to (1.2) on $J^1(\mathbf{R})$ the contact vector field

$$\mathbf{X}_H=p\frac{\partial}{\partial x}-x\frac{\partial}{\partial p}+\left(\frac{p^2}{2}-\frac{x^2}{2}\right)\frac{\partial}{\partial y}\,,\quad \mathcal{L}_{\mathbf{X}_H}\omega=0\,, \tag{1.3}$$

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[1)] *The contact form (0.1) defines the distribution (not-integrable) of the planes which vanish the form. The solution* $l$ *is the integral of this distribution.*

and it induces the contact transformation

$$\begin{pmatrix} x \\ p \\ y \end{pmatrix} \to \begin{pmatrix} \tilde{x}_\tau \\ \tilde{p}_\tau \\ \tilde{y}_\tau \end{pmatrix} ,$$

such that

$$\begin{cases} \frac{d\tilde{x}}{d\tau} = \frac{\partial H}{\partial \tilde{p}} = \tilde{p} \\ \frac{d\tilde{p}}{d\tau} = -\frac{\partial H}{\partial \tilde{x}} = -\tilde{x} \\ \frac{d\tilde{y}}{d\tau} = \left(\frac{\tilde{p}^2}{2} - \frac{\tilde{x}^2}{2}\right) \end{cases} \quad \text{with boundary conditions} \begin{pmatrix} \tilde{x}_\tau \\ \tilde{p}_\tau \\ \tilde{y}_\tau \end{pmatrix} \Big|_{\tau=0} = \begin{pmatrix} x \\ p \\ y \end{pmatrix} .$$

Solving this equation we come to

$$\begin{cases} x_\tau = x\cos\tau + p\sin\tau \\ p_\tau = -x\sin\tau + p\cos\tau \\ y_\tau = y + \frac{1}{4}\left(p^2 - x^2\right)\sin 2\tau + \frac{1}{2}px\left(\cos 2\tau - 1\right) \end{cases}$$

We see that for $\tau = 0$ this is the identity transformation, and for $\tau = \frac{\pi}{2}$ this the Legendre trasnformation

$$\begin{cases} x_{\frac{\pi}{2}} = p \\ p_{\frac{\pi}{2}} = -x \\ y_{\frac{\pi}{2}} = y - px \end{cases}$$

**Application to Clairaut equation**

The Clairaut equation

$$y - xy' = f(y') \tag{2.1}$$

has the one-parametric family of lines

$$y = kx + f(k)\,, \quad k \in \mathbf{R}, -\infty < k < \infty\,, \tag{2.2a}$$

and the special soluction, their envelope: $\varphi(x)$,

$$\varphi(x) = k(x)x + f(k(x)): \quad k(x) \text{ is such that} \frac{\partial\varphi}{\partial k} = x + f'(k) = 0\,. \tag{2.2b}$$

E.g. the solutions of Clairaut equation $y - xy' = {y'}^2$ are the lines $y = kx + k^2$, $k \in \mathbf{R}$ and their envelope, the function $\varphi(x) = -\frac{x^2}{2}$. This is standard.

Apply Legendre transformation (0.3) to Clairaut equation (2.1). The Clairaut equation (2.1) transforms to the algebraic equation

$$\tilde{y} = f(\tilde{x}) \tag{2.3}$$

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The solutions (2.2a), the lines will transform to (generalised) solutions, vertical lines: Every line (2.2a) $l_k = \begin{pmatrix} x = t \\ p = k \\ y = kt + f(k) \end{pmatrix}$ is transformed to vertical line $\begin{cases} \tilde{x} = k \\ \tilde{p} = t \\ \tilde{y} = f(k) \end{cases}$, $-\infty < t < \infty$, and this vertical line is the generalised solution of equation (2.3).

The special envelop solution (2b), the curve

$$l_\varphi = \begin{pmatrix} x = t \\ p = k(t) \\ y = k(t)t + f(k(t)) \end{pmatrix}, \text{where } k(t)\colon t + f(k(t)) = 0$$

is transformed to curve $\begin{cases} \tilde{x} = t \\ \tilde{p} = f'(t) \\ \tilde{y} = f(t) \end{cases}$, which comes from the solution of algebraic equation.

**Remark** it is improtant to note that Legendre transformation is the quasiclassic of Fourier trasnformaation.