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Inversion and stereographic projection

Vek zhivi, vek uchisj,
durakom pomrjoshj

You know how I like stereoraphic projection, it has so much beautiful properties, in particular the property of transforming of rational points to rational ones. Today I learned that stereographic projection is just the restriction of inversion!

In a more details:

Let $\varphi\colon S^n\backslash N \leftrightarrow \mathbf{R}^n$ be a stereographic projection of unit sphere without North pole $N=(0,\dots,0,1)$,

$$S^n\backslash N = \left\{x^i\colon \quad \sum_{i=0}^n x^i x^i = 1, x^0 \neq 1\right\}$$

on the hyperplane

$$\mathbf{R}^n = \left\{x^i = u^i, i=1,\dots,n;\ x^0 = 0\right\}\,.$$

A point $\varphi(A)\in\mathbf{R}^n$ is stereopgraphic image of the point $A\in S^n$ if three points $N, A, \varphi(A)$ are incident (belong to the same line).

This is remarkable map, in particular it has the property:

point $\varphi(A)$ has rational coordinates if and only if the point A has rational coordinates (†)

Now consider the inversion $\hat\varphi$ of $\mathbf{R}^{n+1}$ with respect to the sphere $\tilde S$, such that the centre of this sphere is the north pole N and equatorial circle S^{n-1}_{equator} (points $x^0=0$ on S^n) belongs to this sphere: One can see that the sphere $\tilde S$ is defined by the equation

$$(x^0-1)^2 + \sum_{i=1}^n x^i x^i = 2\,. \qquad (*)$$

The map $\hat\varphi$ has the following appearance:

$$\mathbf{R}^{n+1}\backslash\mathbf{n} \ni \mathbf{x}\ \ \hat\varphi(\mathbf{x}) = \mathbf{n} + \frac{2\mathbf{x}-2\mathbf{n}}{|\mathbf{x}-\mathbf{n}|^2}\,,$$

where $\mathbf{n}$ is a vector attaching origin to the North pole. The image of S^n under inversion $\hat\varphi$ is hyperplane, since $\hat\varphi$ is not define at vector $\mathbf{n}$. It is easy to check also that equatorial circle remains invariant. Using properties of inversion[1)] we come to the

Fact

$$\hat\varphi_{S^n\backslash N} = \varphi\,,$$

i.e. the restriction of inversion with respect to the sphere (*) is stereographic projection φ.

Notice that for inversion $\hat\varphi$ condition (†) holds.

I always was little bit unhappy that there is always a problem when we consider stereographic map φ *in Cartesian coordinates of embedding space. The embedding of* φ *in inversion,* $\hat\varphi$ *makes us little bit happier.*

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1) inversion maps circles and planes to circles and planes