Conic sections and Kepler's law. Newton—Lagrange—Givental

This etude is written on July 2017 on the base of some story happened with me when I was preparing my lectures, and on the base of my conversations with Alexander Givental in May 2014, and his letters to me, in April 2017

§1 **Introduction**

Conic section is an intersection of plane with surface of cone. Let $M\colon k^2x^2+k^2y^2=z^2$ be conic surface, and let $\pi\colon Ax+By+Cz=1$ be a plane. Conic section

$$C\colon \begin{cases} k^2x^2+k^2y^2=z^2 \\ Ax+By+Cz=1 \end{cases} \tag{1.0}$$

1. Conic sections are ellipses or parabolas or hyperbolas, (1.1)

(We do not conisder degenerate case when plane intersects origin.)

Kepler's first law:

trajectories of particle in gravitational field are conic section
In particular orbits of every planet is an ellipse (1.2)
Sun is one of the foci of this ellipse

These two statements are common place for everybody.

Almost everybody who has learnt a bit of higher mathematics knows how to prove statement (1.1) (exercise in Analytical Geometry). Everybody who learnt little bit calculus knows that using the gravitational law, that

$$F=\gamma\frac{mM_{\rm sun}}{R^2} \tag{1.3}$$

one can prove statement (1.2), showing that the solution of differential equation

$$\frac{d^2}{dt^2}\mathbf{R}=\frac{\mathbf{F}}{m}=-\gamma\frac{M_{\rm sun}\mathbf{R}}{R^3}\,, \tag{1.3a}$$

is a conic section(m is a mass of planet, and $M_{\rm sun}$ is Solar mass, γ–gravitational constant). Standard proofs are based on calculus: solving equation (1.3a) in polar coordinates and calculating the elementary integrals we come to conic section [1)] (see calculations in the next paragraph, or in any book on Theoretical Mechanics).

[1)] Usually equation (1.3) is called Newton Graviational Law, and equation (1.3a) Newton Second law. There is an alternative point of view that Robert Hook formulated statement (1.3) in the letter to Newton, and Newton who invented Calculus, deduced Kepler's first law, statement (1.2), solving differential equation (1.3a).

How to come to the statement (1.2) avoiding caclulations of integrals?

Instead conic section (1.0) one can consider its orthogonal projection on the plane $z = 0$. For example if

$$C\colon \begin{cases} k^2x^2 + k^2y^2 = z^2 \\ Ax + By + z = 1 \end{cases}$$

then for its orthogonal projection

$$C_{proj}\colon \begin{cases} k^2x^2 + k^2y^2 = (1 - Ax - By)^2 \\ z = 0 \end{cases} . \tag{1.0}$$

The following statement which is almost evident plays crucial role in this etude:

orthogonal projections of conic sections on the plane is also a conic sections. (4)

In fact

orbits of planets are these projections....

This etude has the following history. Few months ago I was preparing the new lecture for Geometry students about conic sections. I had a task to explain statement (1.1), that sections of conic surface with plane are conic sections (ellipses, parabolas or hyperbolas). The problem was that I could not find a beautiful and short explanation of the statement (1.1) without using extracurricular material. Another trouble was that at that time I had serious problems with eyes, and my access to books was very restricted. Finally I prepared the following explanation: Instead statement (1.1) I can prove to students the statement (1.4), then I can deduced statement (1.1) as the corollary of the statement (1.4). Honeslty considerations looked very bulky, I did not like them, but I had no choice.

I never forget early morning 30 March. In five hours the lecture will begin, where I have to explain to students the statement (1.1) on conic sections. I feel me very unhappy and unsatisfied with the way how I want to deliver the lecture, since the geometry of my considerations on the lecture look very vague. Suddenly I ask me a question: where are the foci of the projected conic section? (The foci of initial conic sections are at the points where Dandellen spheres touch the plane which sects the conic surface.) When the question was put, the answer come almost immediately: It is the vertex of the cone that is one of the foci of projected conic section. This was crucial: I immediately remembered, the May 2014.

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...I am in the Davis University, California, on 80 years celebration
of my teacher Albert S. Schwarz.  I meet there Alexander Givental.
Alexander Givental is famous mathematician, but he is also very much
engaged in teaching mathematics to kids
(see his homepage in Berkeley University).
During conference dinner we are sharing the same table, and Alexander
is explaining me some beautiful properties of conic section.
In particular he is telling me the sentence:
if you consider a projection of ellipse on the horisontal plane,
you come naturally to Kepler's law...
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Recalling this phraze in the morning 30 March 2017, I immediately realised the geometrical meaning of my construction (1.4), which I made preparing the lecture, and I understood what Alexander Givental wanted to tell me three years ago.

Next day I contacted Alexander by e-mail. He immediately sent me the detailed answer.

First, Alexander sent me the article [3]. In this article he gives detailed geometrical explanation why trajectories of planets are conic section, and he does not use the caclulus. You can find this article on the homepage of A.Givental. Thirty years ago, an abstract of this article was included by V. I. Arnold into his joint with V. V. Kozlov and A. I. Neishtadt survey of classical mechanics [2].

In this letter A. Givental also told me that Alain Chensiner noted him that his (A.G.) considerations are very close to Lagrange's proof (see the paper [1]), and Alexander sent me the Lagrange article [1], and the letter where he explained this.

I do not want here to retell the work [3] of A.Givental. (See however the remark at the end of the text.) I just recommend all of you to read this wwonderful paper (it is on his homepage). Here I just try based on the letter of A.Givental to retell the simple proof of statement (1.2) avoiding calculus, and compare this proof with the sandard one. This simple proof can be traced to the work [1] of Lagrange. Of course the work [3] contains complete picture of these considerations.

My modest contribution to this etude is related with the fact that trying to find a simple proof of classical statement (1.1), I realised the importance of the statement (1.4), and on the base of it I try to retell here the constuctions of the papers [1] and [3].

§2 **Standard exlanation of Kepler's Law.**

We will give the standard proof based on the calculus, that trajectory of particle in gravitational filed is a conic section

Let $\mathbf{R} = \mathbf{R}(t)$ is a solution of differential equation (1.3a):

$$\frac{d^2}{dt^2}\mathbf{R} = \frac{\mathbf{F}}{m} = -\gamma \frac{M_{\rm sun}\mathbf{R}}{R^3}\,, \tag{2.1}$$

Consider the angular momentum

$$\mathbf{M} = m\dot{\mathbf{R}} \times \mathbf{R} \qquad (2.2)\,.$$

It is preserved: $\dot{M} = m\dot{\mathbf{R}} \times \dot{\mathbf{R}} + m\ddot{\mathbf{R}} \times \mathbf{R} = 0 + \left(-\gamma \frac{mM_{\rm sun}\mathbf{R}}{R^3}\right) \times \mathbf{R} = 0$.

Vector $\mathbf{R}(t)$ is orthogonal to the constant vector $\mathbf{M}$. Consider Cartesian coordinates x, y, z such that $\mathbf{M}$ is directed along axis OZ, and $\mathbf{R}(t) = x(t)\mathbf{e}_x + y(t)\mathbf{e}_y$ belongs to the plane OXY:

$$\mathbf{R}(t) = x(t)\mathbf{e}_x + y(t)\mathbf{e}_y\,, \quad \mathbf{M}(t) = \mu\mathbf{e}_z\,, (\mu \text{ is constant})\,. \tag{2.3}$$

Let φ, r be polar coordinates in the plane OXY, $x = r\cos\varphi, y = r\sin\varphi$.

Notice that in polar coordinates

$$\mu = m(x\dot{y} - y\dot{x}) = r\cos\varphi(\dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi) - r\sin\varphi(\dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi) = mr^2\dot{\varphi}\,, \tag{2.4}$$

This is integral of motion.

Equations (2.1) imply that for coordinate $r = \sqrt{x^2 + y^2}$

$$\ddot{r} = \frac{d}{dt}\dot{r} = \frac{d}{dt}\left(\frac{x\dot{x} + y\dot{y}}{r}\right) = \frac{x\ddot{x} + y\ddot{y}}{r} + \frac{\dot{x}^2 + \dot{y}^2}{r} - \frac{(x\dot{x} + y\dot{y})^2}{r^3} =$$

$$\frac{x\left(-\gamma\frac{M_{\rm sun}x}{r^3}\right) + y\left(-\gamma\frac{M_{\rm sun}y}{r^3}\right)}{r} + \frac{(\dot{x}^2 + \dot{y}^2)(x^2 + y^2) - (x\dot{x} + y\dot{y})^2}{r^3} =$$

$$-\gamma\frac{M_{\rm sun}}{r^2} + \frac{(x\dot{y} - y\dot{x})^2}{r^3} = -\gamma\frac{M_{sun}}{r^2} + \frac{\mu^2}{m^2r^3}\,, \tag{2.5}$$

i.e.

$$\frac{d}{dt}\left(\frac{\dot{r}^2}{2}\right) = \dot{r}\ddot{r} = -\gamma\frac{M_{\rm sun}}{r^2}\dot{r} + \frac{\mu^2}{m^2r^3}\dot{r} = \frac{d}{dt}\left(\gamma\frac{M}{r} - \frac{\mu^2}{2m^2r^2}\right)\,.$$

Thus we come to the second integral of motion:

$$\frac{d}{dt}\left(\frac{m\dot{r}^2}{2}-\gamma\frac{mM_{\text{sun}}}{r}+\frac{\mu^2}{2mr^2}\right)=0\Rightarrow\frac{m\dot{r}^2}{2}-\gamma\frac{mM_{\text{sun}}}{r}+\frac{\mu^2}{2mr^2}=E\,. \qquad (2.6)$$

We come to first order differential equations [2]:

$$\begin{cases}\frac{dr(t)}{dt}=\sqrt{\frac{2}{m}}\sqrt{E+\gamma\frac{mM}{r}-\frac{\mu^2}{2mr^2}}\\ \frac{d\varphi(t)}{dt}=\frac{\mu}{mr^2}\end{cases}\Rightarrow\frac{d\varphi}{dr}=\frac{\frac{\mu}{r^2}}{\sqrt{2mE+\gamma\frac{2mM_{\text{sun}}}{r}-\frac{\mu^2}{r^2}}} \qquad (2.7)$$

thus

$$\varphi=\int\frac{\frac{\mu dr}{r^2}}{\sqrt{2mE+\gamma\frac{2m^2M_{\text{sun}}}{r}-\frac{\mu^2}{r^2}}} \qquad (2.8)$$

Integrating we come to

$$\int\frac{\frac{\mu dr}{r^2}}{\sqrt{2mE+\gamma\frac{2m^2M_{\text{sun}}}{r}-\frac{\mu^2}{r^2}}},=-\int\frac{\mu d\left(\frac{1}{r}\right)}{\sqrt{\gamma^2\frac{m^2M_{\text{sun}}^2}{\mu^2}+2mE-\mu^2\left(\frac{1}{r}-\gamma\frac{m^2M_{\text{sun}}}{\mu^2}\right)^2}}=$$

$$\arcsin\left\{\left(\frac{\mu}{\sqrt{\gamma^2\frac{m^2M_{\text{sun}}^2}{\mu^2}+2mE}}\right)\left(\frac{1}{r}-\gamma\frac{m^2M_{\text{sun}}}{\mu^2}\right)\right\} \qquad (2.9)$$

i.e.

$$\sin\varphi=\frac{\mu}{\sqrt{\gamma^2\frac{m^2M_{\text{sun}}^2}{\mu^2}+2mE}}\left(\frac{1}{r}-\gamma\frac{m^2M_{\text{sun}}}{\mu^2}\right)=\frac{1}{\sqrt{1+\frac{2E\mu^2}{\gamma^2M_{\text{sun}}^2m^3}}}\left(\frac{\mu^2}{\gamma m^2M_{\text{sun}}}\frac{1}{r}-1\right) \qquad (2.9b)$$

Introduce notations:

$$e=\sqrt{1+\frac{2E\mu^2}{\gamma^2M_{\text{sun}}^2m^3}}\,,k=\frac{\mu^2}{\gamma m^2M_{\text{sun}}}\,,$$

then (App1.9b) has appearance:

$$e\sin\varphi=\frac{k}{r}-1 \qquad (2.10)$$

This is conic section with excentricet e, and one of the foci at the origin.

Remark `in fact it has to be` $\cos\varphi$`, not` $\sin\varphi$

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[2] $E=\frac{m\dot{r}^2}{2}-\gamma\frac{mM_{\text{sun}}}{r}+\frac{\mu^2}{2mr^2}$ is energy, $\frac{m\dot{r}^2}{2}$ is kinetik enery, and $U_{\text{eff}}=-\gamma\frac{mM_{\text{sun}}}{r}+\frac{\mu^2}{2mr^2}$ is effective potential energy

§3 **Simple exlanation of Kepler's Law (avoiding calculations of integrals)**

In the previous paragraph we performed geometrical considerations (equations (2.1)—(2.5)). On the base of these equations we formulated first order differential equations (2.7), solved these differential equations using calculus (see (2.8), (2.9)) and come to equation (2.10) of conic section.

Now we will return to the considerations (2.1)—(2.5) of the previous paragraph, and on the base of these considerations we will come to equation (2.10) avoiding the differential equations and integrals (2.6)—(2.9). Without calculating explicitly integrals (2.9), (2.9a)) we just will show that the solution $\mathbf{R}(t)$ is a conic section, i.e. it obeys equation (2.10).

Choose an arbitrary solution $\mathbf{R}_{0}(t)$ of equation (2.1). In coordinates (2.3)

$$\mathbf{R}_{0}(t) = x_{0}(t)\mathbf{e}_x + y_{0}(t)\mathbf{e}_y \tag{3.3}$$

We suppose that angular momentum $\mathbf{M} = \mu\mathbf{e}_z \neq 0$.

Writing the equation (2.1) in components we come to

$$\begin{pmatrix} \ddot{x}_{0}\ (t) \\ \ddot{y}_{0}\ (t) \end{pmatrix} = -\gamma \frac{M_{\rm sun}}{\left(x_{0}^2(t) + y_{0}^2(t)\right)^{\frac{3}{2}}} \begin{pmatrix} x_{0}(t) \\ y_{0}(t) \end{pmatrix} . \tag{3.4}$$

Now consider the pair of differential equationts on function $f = f(t)$

1) homogeneous second order linear differential equation

$$\frac{d^2 f}{dt^2} = -\gamma \frac{M_{\rm sun} f}{r_{0}^3(t)} \,, \tag{3.5}$$

and associated with it, non-homogeneous second order linear differential equation

$$\frac{d^2 f}{dt^2} = -\gamma \frac{M_{\rm sun} f}{r_{0}^3(t)} + \frac{\mu^2}{m^2 r_{0}^3(t)} \,, \tag{3.5a}$$

where function $r_{0}(t)$ is defined by solutions $x_{0}(t)$, $y_{0}(t)$ (3.4):

$$r_{0}(t) = \sqrt{x_{0}^2(t) + y_{0}^2(t)} \,. \tag{3.6}$$

Notice that $x_{0}(t)$, $y_{0}(t)$ are two independent solutions of second order homogeneous equation (we suppose that angular momentum $\mu = x\dot{y} - y\dot{x}$ does not vanish),].

Notice also that equation (2.5) implies that function (3.6) is the solution of non-homogeneous equation (3.5a). Indeed due to equation (2.5) we have:

$$\frac{d^2 r_{0}(t)}{dt^2} = \frac{d^2}{dt^2}\left(\sqrt{x_{0}^2(t) + y_{0}^2(t)}\right) = -\gamma \frac{M_{\rm sun}}{r_{0}^2(t)} + \frac{\mu^2}{m^2 r_{0}^3(t)} = .-\gamma \frac{M_{\rm sun} r_{0}(t)}{r_{0}^3(t)} + \frac{\mu^2}{m^2 r_{0}^3(t)} \,.$$

Non-homogeneous equation (3.5a) has also the following trivial solution: constant function

$$k = g_0 = \frac{\mu^2}{\gamma m^2 M_{\rm sun}}\,. \tag{3.7}$$

The difference of solutions (3.6) and (3.7) is a solution of homogeneous equation (3.5). Thus we come to the equation:

$$r_{_0}(t) - g_0 = Ax_{_0}(t) + By_{_0}(t)\,. \tag{3.8}$$

where A, B are constants, i.e.

$$r_{_0}(t) = k + Ar_{_0}(t)\cos\varphi_{_0}(t) + Br_{_0}(t)\sin\varphi_{_0}(t) = k + er_{_0}(t)\sin(\varphi_{_0}(t)+\delta)\,,\quad e = \sqrt{A^2+B^2}\,, \tan\delta = \frac{B}{A} \tag{3.8}$$

Thus $r_{_0}(t), \varphi_{_0}(t)$ obeys equation (2.10) for conic with focus at the origin,

Notice that the curve (3.9) is the projection of conic section

$$C\colon \quad \begin{cases} x^2 + y^2 = z^2 \\ z = Ax + By \end{cases}$$

on the plane $z = -k$.

Remark[3]. Geometrical meaning of considerations (3.5)-(3.8) is the following: Consider

$$\mathbf{R}(t) = \mathbf{R}_{_0}(t) + (r_{_0}(t) - g_{_0})\mathbf{e}_z$$

Then previous calculations imply that $\mathbf{R}(t)$ also obeys equation (2.1). 'momentum' $\mathbf{R}\times\dot{\mathbf{R}}$ is preserved. We see that $\mathbf{R}(t)$ is intersection of plane with cone surface $x^2 + y^2 - z^2 = 0$, and $\mathbf{r}_{_0}(t)$ its projection, the projection of conic. (See for details [3]).

[1] Lagrange *DES PERTURBATIONS DES COMETES.*— SECTION DDEUXIEME. Integrations des equations differentielles de l'orbite non-altere. pp.419—430, 1785

[2] V. I. Arnold, V. V. Kozlov, A. I. Neishtadt *Mathematical aspects of classical and celestial mechanics.* Dynamical systems III (Encyclopaedia of Mathematical Sciences), Springer, 1987.

[3] Alexander Givental *Keplers Laws and Conic Sections* Arnold Mathematical Journal March 2016, Volume 2, Issue 1, pp 139148

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