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Distance in Lobachevsky geometry

§ 0

Let $z = x_1+ix_2, w = x_2+iy_2$ be two points in the upper half plane of the complex plane $\mathbf{C}$, $(y_1, y_2 > 0.)$ In this etude-exercise we will derive explicitl formula for the hyperbolic distance between arbitrary two points:

$$d(z,w) = \operatorname{arccosh}\left(1 + \frac{(x_2-x_1)^2+(y_2-y_1)^2}{2y_1y_2}\right). \tag{0.0}$$

Recall that upper half plane of $\mathbf{C}$ with metric

$$G = \frac{dx^2+dy^2}{y^2} \tag{0.1}$$

is a model of Lobachevsky plane.

Geodesics in this model are vertical lines and upper half-circles with centre on the axis OX.

In the special case if the points A, B are on the same vertical line, $A = z = x_1 + iy_1$ and $B = w = x_2 + iy_2$ with $x_1 = x_2$ then one can easy to calculate the distance = length of the arc of geodesic (segment of vertical line): $\begin{cases} x(t) = x_1 \\ y(t) = t \end{cases}, y_1 < t < y_2$c between these points:

$$d(z,w)\big|_{x_1=x_2} = \int_{t_1}^{t_2}\sqrt{\frac{x_t^2+y_t^2}{y^2(t)}}dt = \int_{t_1}^{t_2}\frac{dt}{t} = \left|\log\frac{y_1}{y_2}\right|, \tag{0.2}$$

(Compare with (0.0)).

In the general case if points z, w have not the same ordinate, then the arc of geodesic between these points is the arc of the circle with centre on the OX axis passing trough these points:

$$\begin{cases} x(\varphi) = a + R\cos\varphi \\ y(\varphi) = R\sin\varphi \end{cases}, \qquad \varphi_1 \le \varphi \le \varphi_2$$

and

$$d(z,w)\big|_{x_1\neq x_2} = \int_{\varphi_1}^{\varphi_2}\sqrt{\frac{x_t^2+y_t^2}{y^2(t)}}dt = \int_{\varphi_1}^{\varphi_2}\frac{d\varphi}{\sin\varphi} = \log\tan\frac{\varphi}{2}\Big|_{\varphi_1}^{\varphi_2} = \left|\log\left(\frac{\tan\frac{\varphi_1}{2}}{\tan\frac{\varphi_1}{2}}\right)\right| \tag{0.3}$$

One can write the explicit formula for arbitrary points using

a) equation (0.3): One has to express 'polar coordinates' in terms of Cartesian

b) using formula (0.1) and invariance of distance with respect to Mobius transformations

c) express the length in terms of cross-ratio, which is Mobius invariant.

In the first paragraph we will come to formula (0.0) using the invariance of distance.

Then we will suggest another calculation using the explicit expression (0.3) and at the end we will consider the calculation using cross-ratio.

§ 1 Calculation using invariance

First we consider the case if a point $B = w = i$. Mobius Transformation

$$z \mapsto \frac{z+a}{1-az}\,,\,, a \in \mathbf{R}$$

preserves the point $w = i$. Find a parameter a such that this transformation transforms the point z to the point $z' = iy'$. Then formula (0.2) implies that

$$d(z,w) = d(z,i) = d(z',i) = |\log|z'||\ .$$

Find the required parameter a: we have to find a such that $z' = \frac{z+a}{1-az}$ is also on OY axis, i.e.

$$a\colon\,, \operatorname{Re}\frac{z+a}{1-az} = 0\,,$$

i.e.

$$\frac{z+a}{1-az} + \frac{\bar{z}+a}{1-az} = \frac{(z+a)(1-a\bar{z})+(\bar{z}+a)(1-az)}{(1-az(1-a\bar{z})} = 0\,.$$

We come to the quadratic equation on a

$$(z+a)(1-a\bar{z})+(\bar{z}+a)(1-az) = (z+\bar{z}) + 2a(1-z\bar{z}) - a^2(z+\bar{z}) = 0$$

which defines Mobius transformation which put two points on the axis $0Y$, provided the first point $w = i$. Solving this equation we come to

$$a = \frac{1-z\bar{z} \pm \sqrt{(1-z\bar{z})^2+(z+\bar{z})^2}}{z+\bar{z}} = \frac{1-z\bar{z} \pm \sqrt{1+|z|^4+z^2+\bar{z}^2}}{z+\bar{z}}$$

i.e.

$$z' = iy' = \frac{z+a}{1-az} = \frac{z + \frac{1-z\bar{z}+\sqrt{1+|z|^4+z^2+\bar{z}^2}}{z+\bar{z}}}{1 - \frac{1-z\bar{z}+\sqrt{1+|z|^4+z^2+\bar{z}^2}}{z+\bar{z}}z} =$$

$$\frac{1+z^2+\sqrt{1+|z|^4+z^2+\bar{z}^2}}{\bar{z}(1+z^2) - z\sqrt{1+|z|^4+z^2+\bar{z}^2}} = \frac{1+z^2+\sqrt{1+|z|^4+z^2+\bar{z}^2}}{\bar{z} - z\left(\sqrt{1+|z|^4+z^2+\bar{z}^2} - |z|^2\right)} \tag{1.1}$$

transforms an arbitrary point $z = x+iy$ with $y \neq 0$ to the point on the axis OY. (we choose the root with $'+'$, the second root is for the points on OX but we are not interested in these points.)

In Cartesian coordinates we have

$$z' = iy' = \frac{1 + x^2 - y^2 + S(x,y) + 2xyi}{x(1 + x^2 + y^2 - S(x,y)) - iy(1 + S(x,y) - x^2 - y^2)}\,, \tag{1.2}$$

where we denote

$$S(x,y) = \sqrt{1 + (x^2 + y^2)^2 + 2x^2 - 2y^2}\,. \tag{1.3}$$

The choice of the parameter a above provides that z' in equation (1.3) is on OY axis, i.e.

$$(1 + x^2 - y^2 + S + 2ixy)(1 + x^2 + y^2 - S) = 2xy^2(1 + S - x^2 - y^2)\,.$$

Now we "play" with formula (1.2). Use the following simple fact:
`Fact` If $z' = iy' = \frac{a+bi}{c-di}$, i.e. $ac = bd$, then

$$z' = iy' = \frac{a + bi}{c - di} = \frac{i(b - ia)}{c - di} = i\frac{b}{c} = i\frac{a}{d} \tag{1.4}$$

According to this identity return to formula (1.2):

$$z' = iy' = \frac{1 + z^2 + \sqrt{1 + |z|^4 + z^2 + \bar{z}^2}}{\bar{z} - z\left(\sqrt{1 + |z|^4 + z^2 + \bar{z}^2} - |z|^2\right)} =$$

$$\frac{\overbrace{(1 + x^2 - y^2 + S)}^{a} + \overbrace{2xy}^{b}\,i}{\underbrace{x(1 + x^2 + y^2 - S)}_{c} - i\underbrace{y(1 + S - x^2 - y^2)}_{d}} = \frac{1 + x^2 - y^2 + S}{y(1 - x^2 - y^2 + S)}i = \frac{2y}{1 + x^2 + y^2 - S}i\,,$$

We see that

$$y' = \frac{1 + x^2 - y^2 + S}{y(1 - x^2 - y^2 + S)} = \frac{1 + x^2 - y^2 + \sqrt{(1 + x^2 - y^2)^2 + 4x^2y^2}}{y\left(1 - x^2 - y^2 + \sqrt{(1 - x^2 - y^2)^2 + 4x^2}\right)} =$$

$$\frac{2xy\left(\left(\frac{1+x^2-y^2}{2xy}\right) + \sqrt{\left(\frac{1+x^2-y^2}{2xy}\right)^2 + 1}\right)}{2xy\left(\left(\frac{1-x^2-y^2}{2x}\right) + \sqrt{\left(\frac{1-x^2-y^2}{2x}\right)^2 + 1}\right)} = e^{\operatorname{arcsh}\frac{1+x^2-y^2}{2xy} - \operatorname{arcsh}\frac{1-x^2-y^2}{2x}} \tag{1.5}$$

Note that

$$ch\,(a - b) = \sqrt{1 + a^2}\sqrt{1 + b^2} - ab$$

i.e.

$$ch\left(\operatorname{arcsh}\frac{1 + x^2 - y^2}{2xy} - \operatorname{arcsh}\frac{1 - x^2 - y^2}{2x}\right) =$$

$$\sqrt{1+\left(\frac{1+x^2-y^2}{2xy}\right)^2}\sqrt{1+\left(\frac{1-x^2-y^2}{2x}\right)^2}-\frac{1+x^2-y^2}{2xy}\frac{1-x^2-y^2}{2x}=$$

$$\frac{S^2(x,y)-(1+x^2-y^2)(1-x^2-y^2)}{2x^2y}=\frac{1+(x^2+y^2)^2+2x^2-2y^2-(1-y^2)^2+x^4}{2x^2y}=$$

$$\frac{x^2+y^2+1}{2y}\,.$$

Hence we see that in equation (1.5)

$$\log y'=\operatorname{arcsh}\frac{1+x^2-y^2}{2xy}-\operatorname{arcsh}\frac{1-x^2-y^2}{2x}=\operatorname{arccosh}\left(\frac{x^2+y^2+1}{2y}\right)\,.$$

Now using invariance of the distance with respect to translations $\begin{cases} x\mapsto x+a \\ y\mapsto y\end{cases}$ and dilations $\begin{cases} x\mapsto \lambda x \\ y\mapsto \lambda y\end{cases}$ we come to the final formula:

$$`ad(z,w)=d(x_1+iy_1,x_2+iy_2)=d\left(x_1-x_2+iy_1,iy_2\right)=d\left(\frac{x_1-x_2+iy_1}{y_2},i\right)=$$

$$\operatorname{arccosh}\left(\frac{x^2+y^2+1}{2y}\right)\Big|_{x\mapsto\frac{x_1-x_2}{y_2},y\mapsto\frac{y_1}{y_2}}=\operatorname{arccosh}\left(\frac{\left(\frac{x_1-x_2}{y_2}\right)^2+\left(\frac{y_1}{y_2}\right)^2+1}{2\left(\frac{y_1}{y_2}\right)}\right)=$$

$$\operatorname{arccosh}\left(\frac{(x_1-x_2)^2+y_1^2+y_2^2}{2y_1y_2}\right)=\operatorname{arccosh}\left(1+\frac{(x_1-x_2)^2+(y_1-y_2)^2}{2y_1y_2}\right)\,.$$

This is what we wanted.

We calculated using invariance properties, but our calculations still are brute force calculations. Try to do it in another way:

§ 2 Straightforward calculations using formula (0.2)

Let $A=z_1=x_1+iy_1$, $B=z_2=x_2+iy_2$ be two points. To use equation (0.2) we have to calculate parameters a,R,φ_1,φ_2. If $z_A=x_1+iy_1$, $z_B=x_2+iy_2$, then arc of geodesic is $(x-a)^2+y^2=R^2$ and solving equation

$$(x_1-a)^2+y_1^2=(x_2-a)^2+y_2^2$$

we come to

$$a=\frac{x_1+x_2}{2}+\frac{y_1^2-y_2^2}{2(x_1-x_2)}\,,\ \text{and}\quad R=\sqrt{(x-a)^2+y^2}=$$

$$\sqrt{\left(x_1 - \frac{x_1+x_2}{2} - \frac{y_1^2-y_2^2}{2(x_1-x_2)}\right)^2 + y_1^2} = \sqrt{\frac{(x_1-x_2)^2}{4} + \frac{(y_1^2-y_2^2)^2}{4(x_1-x_2)^2} + \frac{y_1^2+y_2^2}{2}}\,, \quad (2.1)$$

Thus we can calculate φ_1, φ_2 and we will come to the formula for the distance....

§ 3 **Cross-ratio and distance**

For two arbitrary points A, B, the distance is the length of of the arc of geodesic (the half-circle with centre on OX axis) passing through these points.

Consider points P_∞, Q_0 such that these points are ending and beginning points of the half-circle (the half-circle with centre on OX axis) passing through these points. it is easy to see that formula

$$d(z,w) = |\log(A, B, P_\infty, Q_0)| = \left|\frac{\log\left((z_A - z_{P_\infty})(z_B - z_{Q_0})\right|}{(z_A - z_{Q_0})(z_B - z_{P_\infty})}\right| \quad (3.1)$$

really defines the hyperbolic length. Indeed it is obviously invariant with respect to Mobius transformations, and in the case if points A, B have the same x-coordinate, this formula is coincided with equation (0.2):

$$z_A = x_1 + iy_1, z_B = x_2 + iy_2, x_1, = x_2 \Rightarrow P_\infty = \infty, Q_0 = 0$$

and equation (2.1) is reduced to

$$l = |\log(A, B, \infty, 0)| = \left|\log\left(\frac{(z_A - \infty)(z_B - 0)}{(z_A - 0)(z_B - \infty)}\right)\right| = \left|\log\frac{y_2}{y_1}\right|.$$

To perform the calculations in equation (2.1) we have to use calculations (2.1) in the previous paragraph.

If $z_A = x_1 + iy_1$, $z_B = x_2 + iy_2$, then arc of geodesic is $(x-a)^2 + y^2 = R^2$ and solving equation

$$(x_1 - a)^2 + y_1^2 = (x_2 - a)^2 + y_2^2$$

we come to

$$a = \frac{x_1+x_2}{2} + \frac{y_1^2-y_2^2}{2(x_1-x_2)}\,, \text{ and } \quad R = \sqrt{(x-a)^2+y^2} =$$

$$\sqrt{\left(x_1 - \frac{x_1+x_2}{2} - \frac{y_1^2-y_2^2}{2(x_1-x_2)}\right)^2 + y_1^2} = \sqrt{\frac{(x_1-x_2)^2}{4} + \frac{(y_1^2-y_2^2)^2}{4(x_1-x_2)^2} + \frac{y_1^2+y_2^2}{2}}\,,$$

and

$$P_\infty = a - R, \, , Q_0 = a + R\,.$$