Minimal formulation of localisation principle

`10 October, Cyprus---12 October David's birthday.`

Here I try to formulate the statement on localisation which seems to be very "rational". It is based on my notes in 2013, and these notes are based on understanding of Belavin calculations based on the classic paper of Schwarz

Let $F(x) = e^{iH(x)}$ be an arbitrary (non-zero valued) function in symplectic manifold $(M\,,\Omega)$. Consider the integral

$$\int_M e^{iH} dpdq\,, \tag{1}$$

where $dpdq$ is Lioville measure in M:

$$dpdq = \underbrace{\Omega \wedge \ldots \wedge \Omega}_{n\text{-times}}, \quad 2n\text{=dimension of } M\,.$$

Statement The integral (1) can be localised if there exist $D_{\mathbf{H}}$-invariant 1-form ω which produces symplectic structure on M (i.e. $\tilde{\Omega} = d\omega$ is non-degenerate 2-form).

We denote by $\mathbf{D}_H = K$ Hamiltonian vector field of Hamiltonian H

$$\Omega \rfloor \mathbf{K} + dH\,(\mathbf{K}) = 0\,,\,,(\mathbf{K} = D_H)\,,\quad \mathcal{L}_{\mathbf{K}}\omega = 0\,.$$

In the case if $\{x_i\}$ are points where vector field $\mathbf{K} = D_{\mathbf{H}}$ vanishes,then this vector field defines at every point the linear operator dK_i* This localised integral is equal (up to coefficients containing π) to

$$\frac{1}{n!}\sum_{x_i} \frac{e^{iH(x_i)}}{\sqrt{\det dK(x_i)}} \tag{1a}$$

Namely following Belavin ** consider a function

$$Z(t) = \int_{\Pi TM} e^{i\left(H+\Omega+t\sqrt{\mathcal{L}_{\mathbf{K}}\omega}\right)} dpdqd\xi = \int_{\Pi TM} e^{i\left(H+\Omega+t\left(h+\tilde{\Omega}\right)\right)} dpdqd\xi\,, \tag{2}$$

where ΠTM is tangent bundle with reversed parity of fibres,$dpdqd\xi$ is canonical measure on ΠTM, h is an 'artifical' Hamiltonian defined by the new symplectic structure; this new artificial Hamiltonian produces the same Hamiltonian vector fied: $D_h = D_H = \mathbf{K}$.

$$h = \tilde{\Omega} \rfloor \mathbf{K}\,,\quad \tilde{\Omega} = d\omega\,.$$

---

* the action of operator dK_i on an arbitrary tangent vector $\mathbf{v} \in T_{x_i}M$ can be defined by an equation $dK_i(\mathbf{v}) = [\tilde{\mathbf{v}}, K]$, where $\tilde{\mathbf{v}}$ is an arbitrary vector field such that its value at the point x_i is equal to $\mathbf{v}$, and $[\,,\,]$ is the commutator of vector fields. The vanishing of ector K at the point x_i provides the correctness of this definition.

** I did the slight but important modification of his calculations

We use notation $\sqrt{\mathcal{L}_{\mathbf{K}}}$ for equivariant differential:

$$\sqrt{\mathcal{L}_{\mathbf{K}}}\omega = d_K\omega = d\omega + \omega\rfloor\mathbf{K}\,.$$

Remark We denote by the same letter differential form and the function on ΠTM corresponding to this form.

Lemma A function (2) `does not depend` on t:

We use this lemma to reduce the calculation of initial integral $Z(0)$ to the integral $Z(T)$, which for big t can be expressed in terms of artificial HHamiltonian h. Lemma implies that the initial integral can be calculated in terms of an artificial hamiltonian, and this integral may be caluclated using the stationary phase method:

$$\int e^{iH}dpdq = Z(0) = Z(\infty)\,.$$

i.e.

$$\int_M e^{iH}dpdq = \int_M e^{iH}\underbrace{\Omega\wedge\ldots\wedge\Omega}_{n\text{-times}} = \int_{\Pi TM} e^{i(H+\Omega)}dpdqd\xi =$$

$$\lim_{T\to\infty}\int_{\Pi TM} e^{i\left(H+\Omega+T\sqrt{\mathcal{L}_{\mathbf{K}}\omega}\right)}dpdqd\xi = \lim_{T\to\infty}\int_{\Pi TM} e^{i\left(H+\Omega+T\left(h+\tilde\Omega\right)\right)}dpdqd\xi\,, \qquad (3)$$

Now notice that for every T

$$\int_{\Pi TM} e^{i\left(H+\Omega+T\left(h+\tilde\Omega\right)\right)}dpdqd\xi = \int_M e^{i(H+Th)}\sum_{k=0}^{n}\frac{T^k\tilde\Omega^k\wedge\Omega^{n-k}}{k!(n-k)!}$$

and for every term $\int_{\Pi TM} e^{i\left(H+\Omega+T\left(h+\tilde\Omega\right)\right)}dpdqd\xi = \int_M e^{i(H+Th)}\sum_{k=0}^{n}\tilde\Omega^k\wedge\Omega^{n-k}$ the integral is proportional to t^{k-n}, i.e. it is tending to zero if $T\to\infty$ and $k\neq n$. Hence we have that in equation (3)

$$Z(0) = Z(\infty) = \lim_{T\to\infty}\int_{\Pi TM} e^{i\left(H+\Omega+T\left(h+\tilde\Omega\right)\right)}dpdqd\xi = \lim_{T\to\infty}\int_M e^{i(H+Th)}\frac{T^n\tilde\Omega^n}{n!}\,.$$

Notice that at the points $\{x_i\}$ where vector field $\mathbf{K}$ vanishes, artificial Hamiltonian $h=\omega\rfloor\mathbf{K}$ and its first derivatives vanish also:

$$h\big|_{x=x_i} = \omega_i K^i\big|_{x=x_i} = 0\,, \frac{\partial h}{\partial x^k}\big|_{x=x_i} = \tilde\Omega_{km}K^m\big|_{x=x_i} = 0\,. \qquad (4)$$

Hence calculating the last integral using the stationary phase method we come to

$$\lim_{T\to\infty}\left[\frac{T^n}{n!}\sum_{x_i} e^{iH(x_i)}\int_M e^{iT\left(h_{pq}(x_i)(x^p-x_i^p)(x^q-x_i^q)+\ldots\right)}\tilde\Omega^n\right] =$$

$$\frac{1}{n!}\sum_{x_i} e^{iH(x_i)}\sqrt{\frac{\det\tilde{\Omega}}{\det Hessian \text{ of } h}}\Big|_{x_i}\,. \tag{5}$$

Using equation (4) we see that for Hessian of h at stationary points

$$\frac{\partial^2 h}{\partial x^p \partial x^q}\Big|_{x_i} = \tilde{\Omega}_{pr}\frac{\partial K^r}{\partial x^q}\Big|_{x_i}\,,$$

thus at stationary points x_i

$$\det Hessian \text{ of } h\Big|_{x_i} = \det\tilde{\Omega}\cdot\det dK_i.$$

THis means that equation (5) implies the statement (1a).

Finally we prove the lemma.

`Proof of the lemma`

$$\frac{dZ(t)}{dt} = \int_{\Pi TM} it\sqrt{\mathcal{L}_{\mathbf{K}}}\omega e^{i\left(H+\Omega+t\sqrt{\mathcal{L}_{\mathbf{K}}\omega}\right)}dpdqd\xi\,,$$

and due to the $\mathbf{K}$ invariance of the form ω it is equal to

$$\frac{dZ(t)}{dt} = \int_{\Pi TM} it\sqrt{\mathcal{L}_{\mathbf{K}}}\left(\omega e^{i\left(H+\Omega+t\sqrt{\mathcal{L}_{\mathbf{K}}\omega}\right)}\right)dpdqd\xi\,,$$

One can see that the last integral vanishes since $\sqrt{\mathcal{L}_{\mathbf{K}}}\sigma = d\sigma + \sigma\rfloor\mathbf{K}$ Lemma is proved.