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We teach students that condition $\det A = \pm 1$ *still does not mean that the matrix* A *is orthogonal. However if* $\det A = \pm 1$ *and all columns of the matrix* A *have unit length, then this guarantees that* A *is orthogonal matrix: this immediately follows from the fact that volume of parallelepiped* Π *formed by columns of matrix* A *is less or equal to the product of its lengths, and it is equal to the product if and only if all the edges are orthogonal to each other. Sure intuitively this works for any dimensions, and the proof is very simle.*

Theorem Let $A = ||a_{ij}||$, $i, j = 1 \ldots, n$ be $n \times n$ matrix with real entries. Then

$$|\det A| \leq \text{ product of lengths of all its columns} = \prod_{j=1}^{n} \left(\sum_{i=1}^{n} a_{ij} a_{ij} \right) \tag{1a}$$

and

$$|\det A| = \text{ product of lengths of all its columns}$$

if and only if the columns are orthogonal to each other.

If we consider columns of matrix as vectors in $\mathbf{R}^n$, then the statement of theorem means:

$$A = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}, \ |\det A| \leq \prod_{i=1}^{n} |\mathbf{a}_i|, \tag{1a}$$

and

$$|\det A| = \prod_{i=1}^{n} |\mathbf{a}_i|, \Leftrightarrow \text{vectors } \mathbf{a}_i \text{ are orthogonal to each other}$$

In particular we come to the

Corollary Unimodular matrix with columns of unit length is orthogonal matrix.

This theorem has clear geometrically meaning, and it is almost evident for $n \leq 3$:

The volume of parallelepiped (the area of parallelogram) is less than equal to the product of its edges, and it is eual to this product, if and only if all edges are rothogonal to each other.

Proof. For $n = 1, n = 2$ this is obvious: e.g. for $n = 2$ this means that the area of rombus with unit edges is equal to one if and only if the edges are orthogonal, i.e. the rombus is a quadrate.

Consider linear algebra proof for arbitrary n. Let $n \times n$ matrix be presented by n vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$ of $\mathbf{R}^n$:

$$A = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{n-1}, \mathbf{a}_n\}, \mathbf{a}_i \in \mathbf{R}^n .$$

We suppose that these vectors are linearly independent (In the case if these vectors are linearly dependant then $\det A = 0$). Perform Gram-Schmidt procedure:

Consider the vectors $\{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_{n-1}, \mathbf{b}_n\}$ such that

$$\mathbf{b}_1 = \mathbf{a}_1$$
$$\mathbf{b}_2 = \mathbf{a}_2 + \lambda\mathbf{a}_1 \text{, such that } \mathbf{b}_2 \text{ is orthogonal to } \mathbf{b}_1$$
$$\mathbf{b}_3 = \mathbf{a}_3 + \mu_1\mathbf{a}_1 + \mu_2\mathbf{a}_2 \text{, such that } \mathbf{b}_3 \text{ is orthogonal to vectors } \mathbf{a}_1 \text{ and } \mathbf{a}_2$$
$$\mathbf{b}_4 = \mathbf{a}_3 + \nu_1\mathbf{a}_1 + \nu_2\mathbf{a}_2 + \nu_3\mathbf{a}_3 \text{, such that } \mathbf{b}_4 \text{ is orthogonal to vectors } \mathbf{a}_1,\ \mathbf{a}_2 \text{ and } \mathbf{a}_3$$
$$\text{and so on}$$

We come to the new basis $\{\mathbf{b}_i\}$ in $\mathbf{R}^n$ such that all $\mathbf{b}_i$ are pariwise orthogonal to each other:

$$(\mathbf{b}_i, \mathbf{b}_j) = 0\,, \quad \text{if } i \neq j\,,$$

and and for every i

$$\mathbf{a}_i = \mathbf{b}_i + \sum_{k=1}^{i-1} c^m \mathbf{b}_m\,, \quad i = 1, 2, 3, \dots, n\,.$$

This in particular implies that every vector $\mathbf{b}_i$ has the length less than equal to the length of the vector $\mathbf{a}_i$. Moreover It is evident that this orthogonalisation does not change the determinant:

$$\det A = \det\{\mathbf{a}_1, \dots, \mathbf{a}_n\} = \det\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$$

Thus we come to

$$|\det A| = \sqrt{\det A^+ A} = \det \begin{pmatrix} |\mathbf{b}_1|^2 & 0 & 0 & \dots & 0 \\ 0 & |\mathbf{b}_2|^2 & 0 & \dots & 0 \\ 0 & 0 & |\mathbf{b}_3|^2 & \dots & 0 \\ \dots & & & & \\ 0 & 0 & 0 & \dots & |\mathbf{b}_n|^2 \end{pmatrix} =$$

$$= \sqrt{|\mathbf{b}_1|^2 \cdot \dots \cdot |\mathbf{b}_n|^2} \le |\mathbf{a}_1| \cdot \dots \cdot |\mathbf{a}_n|\,.$$

The equalit means that $\mathbf{a}_i = \mathbf{b}_i$.