Conic sections and pursuit problem

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Consider the following pursuit problem:

The point A moves in vertical direction starting at origin with constant speed, and the point B moves from the point (L,0)) with the same speed, such that velocity is directed along the segment BA. One can see that the path of the point B will be assimptotically tending to vertical. What will be the distance between points A and B at $t \to \infty$?

This problem has the following nice but artifical solution.

Denote by θ the angle between the segment $|BA|$ and OY axis. The length of the segment BA is decreasing in time with the speed $v(1 - \cos\theta)$, and the projection of this segment on the axis OY is increasing in time with the same speed $v(1 - \cos\theta)$. Hence the sum of the length of the segment BA and its projection on the axis OY remains invariant in time. At $t = 0$, this sum was equal to L, and If $t \to \infty$, when the segment BA tends to vertical segment, this sum is tending to $2|AB|$, Hence the distance between the points is A and B, the length of the segment BA tending to $\frac{L}{2}$.

Nice and unexpected solution.

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Many many years ago in 1971 when I was in school, I desperately was trying to solve this problem (it was in March 1971 on the Republican Olimpiad in Physics in Yerevan). Later in my student times, I wasy trying to find another solution, which would be not so artificail, but still elegant. Once I realied that conic sections lead to the beaitufil and illuminating solution. Here it is.

Consider the frame of reference such that the point A stands still in this frame. In this frame the point B simultaneously moves to the origin with the same speed it moves vertically down. It means that in this frame of reference, the distance between point B and origin, is the same than the distance between the point B, and the line$y = -L$: this means that the trajectory of the point B in this frame is a PARABOLA with focus at origin, and with directrix $y = -L$. At $t \to \infty$ the point will be at the midpoint of the segment, which is orthogonal to directrix and which is between the origin, the focus, and the directrix, i,e, it will be at the distance $\frac{L}{2}$. $\blacksquare$