Killings of sphere in stereographic projection and Geodesics, (Runge-Lents-Laplace-like vectors)

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Consider sphere of radius 1 in stereographic projection trough North Pole:

$$u = \frac{x}{1-z}, v = \frac{y}{1-z}.$$

The metric is

$$G = \frac{4(du^2 + dv^2)}{(1 + u^2 + v^2)^2}.$$

Volume form of the metric is

$$dv = \det G = \frac{4du \wedge dv}{(1 + u^2 + v^2)^2}.$$

Recall that the nicest way to find Killings, rotations of sphere, it is to consider them as linear Hamiltonians restricted on the sphere ,

$$H = ax + by + cz = a \sin\theta \cos\varphi + b \sin\theta \sin\varphi + c \cos\theta\,,$$

which is provided with symplectic structure $\omega = dv = \sin\theta \wedge d\varphi$. (Symplectic structure on the sphere is the Kirillov symplectic structure on orbit ($= S^2$) in coalgebra $so(3) = E^3$; linear Hamiltonians are elements of this coadjoint algebra.) Do it in stereographic coordinates:

$$H_x = x = \frac{2u}{1 + u^2 + v^2}, H_y = y = \frac{2v}{1 + u^2 + v^2}, H_z = z = \frac{u^2 + v^2 - 1}{1 + u^2 + v^2},$$

Using that $D_H = \omega^{-1} dH$ we come to

$$D_x = \omega^{-1} dH_x = \frac{1 + v^2 - u^2}{2} \partial_v + uv\partial_u\,,$$

$$D_y = \omega^{-1} dH_x = \frac{1 + u^2 - v^2}{2} \partial_u + uv\partial_v$$

$$D_z = \omega^{-1} dH_x = \pm(v\partial_u - u\partial_v)\,.$$

It is representation of $so(3)$:

$$[D_x, D_y] = D_z,\ [D_y.D_z] = D_x,\ [D_z.D_x] = D_y\,.$$

We call them Runge-Lentz since under suitable transformation free particle on sphere becomes particel in Coulomb field and these inegrals are just Runge-Lentz-Laplace -vectors.

It is interesting to find geodesics of sphere (great circles) via these Runge-Lentz vector fields in stereographic coordinates. Of course we know already answer: Image of circles are circles ($u+iv=z\mapsto z'=\frac{Az+B}{-\bar{B}z+\bar{A}}$ is $SU(2)$transformation.)

Great circle passes through two antipodal points: images of antipodal points are points z, z', such that $z'=-\frac{1}{\bar{z}}$. Hence geodesics are circles such that images of antipodal points are edges of their diameters.

But try to claculate them trough geodesics. Geodesics of sphere are equation of motion of Lagrangian

$$L=L_G=G_{ik}\dot{x}^i\dot{x}^k=\frac{2(\dot{u}^2+\dot{v}^2)}{(1+u^2+v^2)^2}\,.$$

Instead solving second order equations consider three integrals of motion $I=K^i\frac{\partial L}{\partial \dot{x}^i}$ corresponding to Killing vector fields $\{D_x, D_y, D_z\}$:

$$I_x=\frac{2}{(1+u^2+v^2)^2}\left(2uv\dot{u}+(1+v^2-u^2)\dot{v}\right)=C_1\,,$$

$$I_y=\frac{2}{(1+u^2+v^2)^2}\left((1+u^2-v^2)\dot{u}+2uv\dot{v}\right)=C_2\,,$$

$$I_z=\frac{2}{(1+u^2+v^2)^2}\left(v\dot{u}-u\dot{v}\right)=C_3\,.$$

We have three integrals of motion which are equal to constants C_1, C_2, C_3 on geodesics (paths of free particle).

Above there are three equations on two unknowns $\dot{u}$ and $\det v$. Hence we come from these three equations on variables $\dot{u}, \dot{v}$ to equation

$$\det\begin{pmatrix} 2uv & 1+v^2-u^2 & C_1 \\ 1+u^2-v^2 & 2uv & C_2 \\ v & -u & C_3 \end{pmatrix}=0$$

(Vectors of integrals of motions belong to the span to two vectors.)

Calculating determinant we come to

$$\det\begin{pmatrix} 2uv & 1+v^2-u^2 & C_1 \\ 1+u^2-v^2 & 2uv & C_2 \\ v & -u & C_3 \end{pmatrix}=(1+u^2+v^2)\left[C_2v-C_1u+C_3(u^2+v^2-1)\right]=0\,,$$

i.e.

$$C_2v-C_1u+C_3(u^2+v^2-1=0\,.$$

These are trajectories of free particle—-geodeisics.

Three cases

1st case: $C_3=0$ we come to straight lines $C_2v-C_1u=0$, images of meridians.

2-nd case: $C_1=C_2=0$ we come to $u^2+v^2=1$, the image of Equator

3-rd case $C_1\neq 0$ or $C_2\neq 0$ and $C_3\neq 0$ we come to circles such that "antipodal points" are inversed. ■