Determinant and Berezinian. Their homological interpretation.

I will give here shortly homological interpretation of (super)determinant. Reading this text you have to do exercises.

First consider determinant.

§*1.* **Determinant**

Let V be linear space. Consider the space $\Pi V \otimes V^*$.

Consider the space C of polynomials on $\Pi V \otimes V^*$. Let $\{\mathbf{e_i}\}$ be an arbitrary basis in V, $\{\mathbf{e^i}\}$ be a dual basis in V^*, and let $\{\mathbf{\Pi e_i}\}$ be corresponding basis in ΠV. We denote by $\{\theta^i\}$ coordinates of vectors in ΠV and by $\{p_i\}$ coordinates of vectors in V^*. in these bases.

Note that $\{\theta^i\}$ are odd.

Polynomials on C are polynomials on θ^i, p_i.

Consider the polynomial

$$Q = \theta^i p_i = \theta^1 p_1 + \ldots + \theta^n p_n \tag{1.1}$$

Exercise 1.1 Show that Q is well-defined, i.e. it does not depend on the choice of basis $\{\mathbf{e}_i\}$ in V.

Exercise 1.2 Show that $Q^2 = 0$.

One can consider Q as a differential on the space C. (This is Koszul-type differential)

Calculate (co)homology of Q:

$$H = Z \backslash B,$$

where Z is space of cocycles: $P \in Z$ if $QP = 0$ and B is space of coboundaries: $P \in P$ if $P = QR$.

Consider the following Laplacian on C:

$$\Delta P(p, \theta) = \frac{\partial^2}{\partial \theta^i \partial p_i} P(p, \theta) \tag{1.2}$$

Exercise 1.3 Calculate anticommutator

$$[\Delta, Q]_+ = \Delta Q + Q \Delta$$

on the space C.

Consider $C = \oplus_{k,r} C_{kr}$ where C_{kr} is the subspace of polynomials which have weight k on θ and weight r on p, i.e.:

$$P \in C_{k,r} \text{ if } \theta^i \frac{\partial}{\partial \theta^i} P = kP,\ p_i \frac{\partial}{\partial p_i} P = rP\,.$$

Exercise 1.4 Calculate $[\Delta, Q]_+$ on subspaces $C_{k,r}$. More exactly show that there exist number $\nu_{k,r}$ such that for an arbitrary $P \in C_{k,r}$

$$[\Delta, Q]_+ P = \nu_{k,r} P$$

Exercise 1.5 Show that all cocycles coboundaries in all subspaces except the subspace $C_{n,0}$, where n-dimension of the vector space V.

Exercise 1.6 Consider cohomology class $c = [\theta^1 \theta^2 \ldots, \theta^n]$. Show that it is not trivial.

Exercise 1.7 Show that every cohomology class is proportional to c, i.e. $H(Q, C) = \mathbf{R}$.

If you did all these exercises then you are ready to see that for an arbitrary linear operator A on V

$$\det A = \frac{[\hat{A}c]}{[c]} \tag{1.!!!}$$

where $[c]$ is an arbitrary non-zero cohomology class, and $\hat{A}c$ the induced action of A on c.

The last formula is cohomological definition of determinant. The pessimist will tell that it is just the translation from "Enlgish (Russian) to French" of the well-known formula

$$\det A = \frac{\omega(A\mathbf{x}_1, A\mathbf{x}_2, \ldots, A\mathbf{x}_{n-1} A\mathbf{x}_n)}{\omega(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{n-1}\mathbf{x}_n)}, \tag{1.!!!??}$$

where ω is an arbitrary n-form , and $\mathbf{x}_1,\dots,\mathbf{x}_n$) is an arbitrary set of n-vectors such that $\omega(\mathbf{x}_1,\dots,\mathbf{x}_n)\neq 0$.

Yes, in some sense pessimist is right, but on the other had the advantage of the formula (1.!!!) is that it can be easily generalised for the supercase opposite to the formula (1.!!!?).

§*1.* Berezinian (superdetermiant)

Now let V be $p|q$-dimensional superspace. Shortly what is it:

Consider Z_2-graded vector space $V_0\oplus V_1$ such that dimension of V_0 equals p and dimension of V_1 equals q.

Let $\{\mathbf{e}_i,\mathbf{f}_\alpha\}$, $i=1,,\dots,p$; $\alpha=1,\dots,q$ be a basis in V. An arbitrary vector $\mathbf{x}$ in V can be decomposed.

$$\mathbf{x}=a^i\mathbf{e}_i+b^\alpha\mathbf{f}_\alpha \tag{2.1}$$

Consider expressions (2.1) where coefficients a^i are even elements and coefficients b^α are odd elements of an arbitrary Grassmann algebra Λ. Thus we come to the set

$$V_\Lambda=\Lambda_0\otimes V_0\oplus\Lambda_1\otimes V_1$$

of Λ-points of the superspace V [1)]

One can consider even linear operators on the super-space V: They are given by K be an even $p|q\times p|q$ matrices

$$K=\begin{pmatrix}K_{00} & K_{01}\\ K_{10} & K_{11}\end{pmatrix}$$

where entries of $p\times p$ matrix K_{00} and $q\times q$ matrix K_{11} are even numbers (even elements of a Grassmann algebra), and entries of $p\times q$ matrix K_{01} and $q\times p$ matrix K_{10} are odd numbers (odd elements of a Grassmann algebra)

Respectively ΠV is $q|p$ dimensional superspace.

Consider the space $\Pi V\otimes V^*$. Coordinates on the space ΠV are $\{\theta^i,y^\alpha\}$ where θ^i are anticommuting (odd elements of Grassmanm algebra) ($i=1,2,\dots,p$) (have parity opposite to the parity of a^i) and y^α are commuting $\alpha=1,\dots,q$ (have parity opposite to the parity of b^α)

Coordinates of the space $\Pi V\otimes V^*$ are $\{\theta^i,y^\alpha,p_i,\varphi_\alpha\}$, where θ^i,φ_α are odd and y^α,p_i are even.

We have

$$Q=\theta^ip_i+y^\alpha\varphi_\alpha. \tag{1.1}$$

Exercise 1.1 Show that Q is well-defined, i.e. it does not depend on the choice of basis $\{\mathbf{e}_i\}$ in V.

Exercise 1.2 Show that $Q^2=0$.

In the same way as for pure bosonic case one can consider Q as a differential on the space C and consider (co)homology of Q:

$$H=Z\backslash B,$$

To calculate the cohomology we consider the same Laplacian:

$$\Delta=\frac{\partial^2}{\partial\theta^i\partial p_i}+\frac{\partial^2}{\partial y^\alpha\partial\varphi_\alpha} \tag{1.2}$$

Exercise In the same way like before calculate $[\Delta,Q]_+=\Delta Q+Q\Delta$ and show that cohomology not vanish only in the one-dimensional subspace of polynomials

$$P=a\theta^1\theta^2\dots,\theta^p\varphi_1\dots\varphi_q$$

Exercise Show that cohomology class $c=[\theta^1\theta^2\dots,\theta^n\varphi_1\dots\varphi_q]$ is not trivial.

Finally we come to Berezinian. Consider a matrix of an arbitrary even operator:

$$K=\begin{pmatrix}K_{00} & K_{01}\\ K_{10} & K_{11}\end{pmatrix}$$

[1)] Superspace is a functor which assigns to an arbitrary Grassmann algebra Λ the set V_Λ of Λ-points.

where entries of $p \times p$ matrix K_{00} and $q \times q$ matrix K_{11} are even numbers (even elements of a Grassmann algebra), and entries of $p \times q$ matrix K_{01} and $q \times p$ matrix K_{10} are odd numbers (odd elements of a Grassmann algebra)

Calculate the action of this operator on cohomology class $c = [\theta^1\theta^2 \dots, \theta^n\varphi_1 \dots \varphi_q]$ we come to the formula:

$$[\hat{K}c] = \mathrm{BerK}[c],$$

where

$$\mathrm{BerK} = \frac{\det\left(K_{00} - K_{10}K_{11}^{-1}K_{01}\right)}{\det K_{11}}$$

Do it!