Projective module of Mobius band global sections

Let C be an algebra of continuous functions on $[0, 2\pi]$. Consider the following two subalgebras of C

$$\Lambda = \{f\colon\ f \in C, f(0) = f(2\pi)\} \tag{1}$$

and

$$P = \{f\colon\ f \in C, f(0) = -f(2\pi)\} \tag{2}$$

P can be considered as module over Λ This module is projective and it is not free. More precisely:

$$P \oplus P = \Lambda \oplus \Lambda \tag{3}$$

Prove (3). Equation (3) means that P is projective (by definition). From (3) it follows that P is not free. Indeed suppose that it is free. Then from embedding (3) it follows that it has one generator f_0. From (2) it follows that f_0 vanishes in some point x_0. Hence f_0 generates submodule of P. Contradiction.

Before proving (3) we note that P is nothing but module of global continuous sections on Mobius band. Condition (3) corresponds to condition that Whithney sum of two Mobius bundles is trivial bundle:

$$M \oplus M = R^2 \times S^1 \tag{4}$$

The proof follows from the following geometrical realization of (4). Consider in R^4 two Mobius bands, bundles over circle:

$$M_1\colon \begin{cases} x = cos\varphi \\ y = sin\varphi \\ z = tcos\frac{\varphi}{2} \\ u = tsin\frac{\varphi}{2} \end{cases}, \qquad M_1\colon \begin{cases} x = cos\varphi \\ y = sin\varphi \\ z = -\tau sin\frac{\varphi}{2} \\ u = \tau cos\frac{\varphi}{2} \end{cases}, \quad 0 \le \varphi \le 2\pi, -\infty < t < \infty, -\infty < \tau < \infty \tag{5}$$

It is evident that this embedding leads to (4). Now from (5) it follows the proof of (3). The isomorphism (3) is given by the formula

$$\rho \begin{pmatrix} t(\varphi) \\ \tau(\varphi) \end{pmatrix} = \hat{T}_{\frac{\varphi}{2}} \begin{pmatrix} z(\varphi) \\ u(\varphi) \end{pmatrix},$$

where $\hat{T}$ is operator of rotation:

$$\hat{T}_{\varphi} = \begin{pmatrix} \cos\varphi, \sin\varphi \\ -\sin\varphi, \cos\varphi \end{pmatrix}$$

To isomorphism (3) corresponds the following projector in $R^2 \times S^1$ on M:

$$\Pi(z, u, \varphi) = \hat{T}_{\frac{\varphi}{2}} \circ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \circ \hat{T}_{\frac{-\varphi}{2}}, \quad 1 - \Pi(z, u, \varphi) = \hat{T}_{\frac{\varphi}{2}} \circ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \circ \hat{T}_{\frac{-\varphi}{2}}$$

It is funny to note that this projector has very simple appearance:

$$\Pi = \frac{1}{2}\left(1 + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \hat{T}_{\varphi}\right)$$