I What is differential form?

Motto: Every ~~H~~ time if we integrate "something" over k-dimensional surface (one dimens. curve, two-dim. surf., ...) we in fact integrate differential form (even if we try not to tell this straightforwardly)

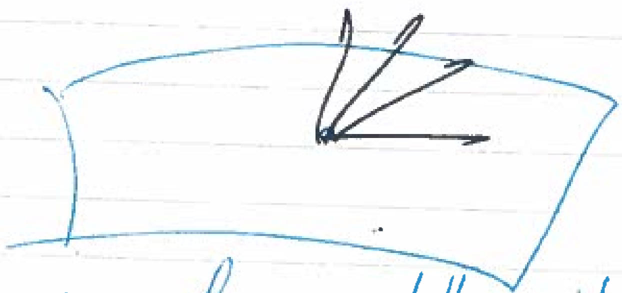
II Language of form allows to avoid problems in curvilinear coordin.

0-form. Function is function on a point, (Definition)

Diff. 1-form is a function on tangent vectors

Ω_1 $\quad \omega = \omega(x, \vec{X})$.

tangent vector has to be attached to the point



$T_{x_0}M$ - vectors tangent to M at the point x_0

one-forms at the point x_0 - $T^*_{x_0}M$ - linear function on tangent vectors

$T^*_{x_0}M$ is vector space dual to vector space $T_{x_0}M$.

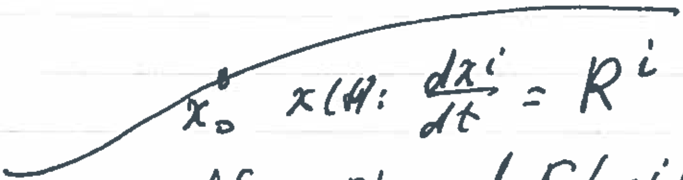
Newton law: $F = ma$ sourse of confusion,

~~Two~~ 0-form — just function on points.

Example of 1-form

Let f be a function

$w(R) = df(x, R)$ = directional derivative of function along vector $R^i \dfrac{\partial f}{\partial x^i} = \partial_{\vec{R}} f$ *



$x(t)$: $\dfrac{dx^i}{dt} = R^i$

$$df(x, R) = \frac{d}{dt} F(x^i(t))\Big|_{t=0} = R^i \frac{\partial f}{\partial x^i}$$

$w = df$ — exact form

Arbitrary form $W = \sum w_i(x)\, dx^i$

(linear combination of exact forms)

$$w(x, R) = \sum w_i(x)\, dx^i(R) = \sum R^i(x) w_i(x) = R^i w_i$$

(sometimes notation is omitted

Not every form is exact, but

Exer. if $w = a\,dx + b\,dy$ is 1-form in $\mathbb{R}^2$ $\exists$ function λ: $\boxed{\lambda w = df}$

* In some books sometimes they 'define'

$$df(x, R) = \underbrace{\nabla f \cdot R}_{\text{even number of mistakes}},$$

One can integrate 1-form over curve (we do not care on coordinates and on any additional structure)

$$\int_C \omega = \int_a^b \omega(\vec{v}(t))\,dt =$$

$$C:\ x^i = x^i(t) \quad a \le t \le b \qquad = \int_a^b \omega_i(x(t))\frac{dx^i(t)}{dt}\,dt.$$

Integral DOES NOT depend on coord. on X

it DOES NOT depend (up to a sign - orientation)

on parameterisation of the curve) *

Sometimes in some books they write

$\int \vec{A}\cdot d\vec{s} = \int \Sigma A^i(x)\frac{dx^i}{dt}$

Vectorfield, scalar product. in fact it is:

$\int g_{ik} A^i(x) dx^k$

g_n

Two-form

Def. Two-form ~~w = w(·~~ is a function depending on <u>two</u> tangent vectors which is antisymmetric

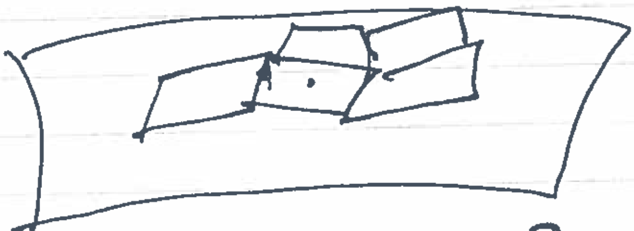
$$w = w(\cancel{x, y}, \vec{X}, \vec{Y}) =$$

$$w(\vec{X}, \vec{Y}) = -w(\vec{Y}, \vec{X})$$

(может быть раньше дать это определение (сразу после определения 1-формы)

Why antisymmetric???

Our aim is to integrate 2-form over surface



$$\int_M w \approx \sum \omega(\xi_i) \text{ Area of } \Pi_{\Delta_i}$$

$$\text{Area of } \Pi_{\Delta_i} = \text{Area of } \Pi_{(\vec{a}, \vec{b})} = w(\vec{a}, \vec{b})$$

for every function on two variables

$$w(\vec{a}, \vec{b}) = \underbrace{w_S(\vec{a}, \vec{b})}_{\frac{w(\vec{a}, \vec{b}) + w(\vec{b}, \vec{a})}{2}} + w_{ANT}(\vec{a}, \vec{b}) \qquad \frac{w(\vec{a}, \vec{b}) - w(\vec{b}, \vec{a})}{2}$$

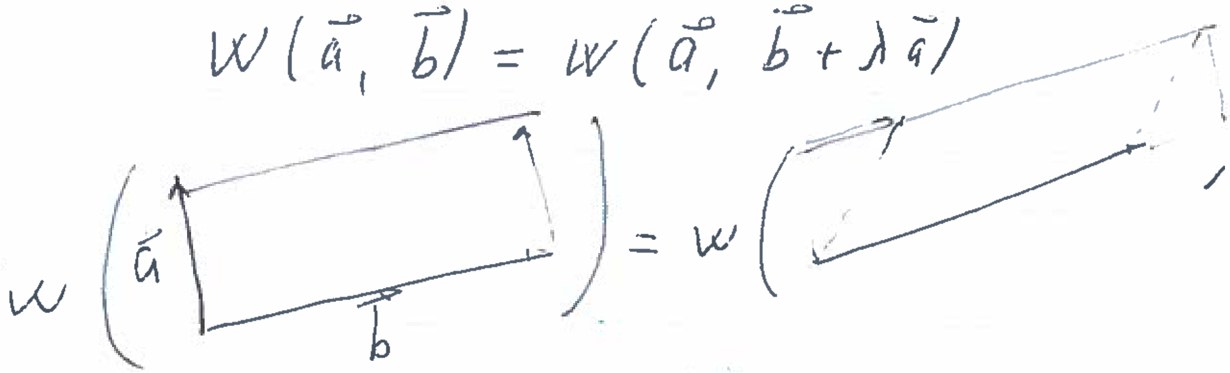
$w(\vec{X}, \lambda\vec{X})$ — has to be equal Zero! $\Rightarrow$

Hence w — MUST to be antisymmetric

Now show that this is enough

$w(\vec{a}, \vec{b}) \sim$ Area of parallelogram $\Pi_{\vec{a}, \vec{b}}$

$$w(\vec{a}, \vec{b}) = w(\vec{a}, \vec{b} + \lambda \vec{a})$$



$\Downarrow$

$$w(\vec{a}, \vec{b}) = C \det \begin{pmatrix} a_x & a_y \\ b_x & b_y \end{pmatrix} = C(a_x b_y - a_y b_x)$$

$dx \wedge dy \stackrel{\text{def.}}{=}$

$$dx \wedge dy (\vec{\partial}_x, \vec{\partial}_y) = 1$$

$$w(a\partial_x + b\partial_y, \; c\partial_x + d\partial_y) =$$

$$= C(ad - bc) \quad w(\partial_x, \partial_y) = ad - bc$$

$$w = C \, dx \wedge dy$$

wedge product
$w = a\,dx + b\,dy$
$\sigma = c\,dx + d\,dy$
$w \wedge \sigma =$
$= (a\,dx + b\,dy) \wedge (c\,dx + d\,dy) =$
$= (ad - bc)\, dx \wedge dy$

$$\int dx \wedge dy = \text{Area of domain}$$

We can perform integr. in arb. coord.;

$$\int dx \wedge dy = \int (x_r dr + x_\varphi d\varphi) \wedge (y_r dr + y_\varphi d\varphi) = \int (x_r y_\varphi - x_\varphi y_r) dr \wedge d\varphi =$$

$$= \int r \, dr \wedge d\varphi \qquad \left[\begin{array}{l} x = r \cos\varphi \\ y = r \sin\varphi \end{array} \right.$$

We use notation for vectors dual to

$\vec{\partial}_x$ attached at the point x_0: $\quad x = x_0 + t, \; y = y_0.$

Wedge product

$(dx \wedge dy)(\vec{a}, \vec{b}) = -w(\vec{b}, \vec{a}) =$

$$\vec{a} \wedge \vec{b} = \frac{1}{2}(\vec{a} \otimes \vec{b} - \vec{b} \otimes \vec{a})$$

w

2-form in

$u^1, \ldots, u^n - u^i$

$$w = \sum w_{ij}(u)\, du^i \wedge du^j$$

$$w(\vec{A}, \vec{B}) = \sum w_{ij}(u)\, du^i \wedge du^j \, (A^m \partial_m, B^n \partial_n)$$

$$= \sum w_{ij} A^i B^j$$

One may introduce

$$\vec{A} \wedge \vec{B} = \frac{1}{2}(A \otimes B - B \otimes A)$$

$$(\vec{A} \wedge \vec{B})^{ij} = \frac{1}{2}(A^i B^j - B^i A^j)$$

$$w(\vec{A}, \vec{B}) = w(\vec{A} \wedge \vec{B})$$

Integral of two-form over surface.

$$\int_M W = \qquad M: \begin{cases} u^i = u^i(\xi, \eta) \\ i = 1, \dots, n. \end{cases}$$

$$= \int_M W_{ij}\, du^i \wedge du^j = \int W_{ij} \left(\frac{\partial u^i}{\partial \xi} d\xi + \frac{\partial u^i}{\partial \eta} d\eta \right) \wedge$$

$$\wedge \left(\frac{\partial u^j}{\partial \xi} d\xi + \frac{\partial u^j}{\partial \eta} d\eta \right) =$$

$$= \int W_{ij} \det \begin{pmatrix} u^i_\xi & u^i_\eta \\ u^j_\xi & u^j_\eta \end{pmatrix} d\xi\, d\eta$$

Example.

Flux of vector field through surface

$$\oint_M \vec{R}\, d\vec{S} = \int_M W_{\vec{R}} =$$

$$W_{\vec{R}}: \quad W_{\vec{R}}(\vec{A}, \vec{B}) = \Omega(\vec{R}, \vec{A}, \vec{B})$$

$=$ (in Cartesian coordinates)

$$\int (R^1\, dy\, dz - R^2\, dx\, dz + R^3\, dx\, dy)$$

In ~~many~~ some books $\oint \vec{R}\, d\vec{S} = \int \vec{R} \cdot d\vec{\sigma} \approx \ldots$

even number of 'mistakes'

Stokes Theorem.

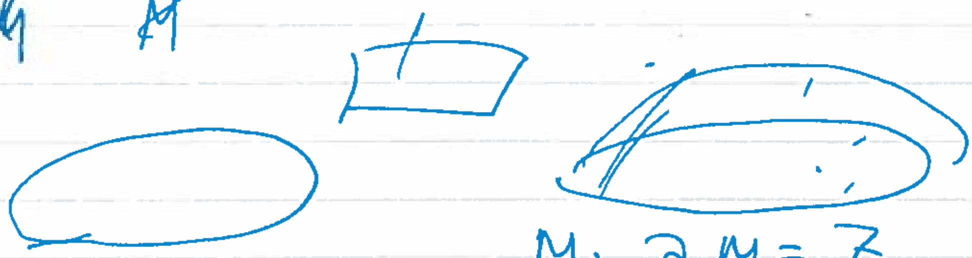
$$\int_C d\omega = \int_{\partial C} \omega$$

This theorem incorporates many theorems

1. $\int_a^b f'(x)\,dx = f(b) - f(a)$

2. $\oint_{\partial D} \vec{R}\, d\vec{S} = \int \operatorname{div} \vec{R}\, dV$ (2)

3. $\cancel{\oint_M}$ $\cancel{\int_M \vec{F}\, d\vec{S}} =$



Z $\quad$ $M: \partial M = Z$

$$\oint_Z \vec{F} \cdot d\vec{s} = \int_M \operatorname{curl} \vec{F} \cdot d\vec{S}$$

Ex. Prove 2

$$\oint \vec{R}\, d\vec{S} = \int_{\partial D} W_{\vec{R}} = \int_{\partial D} \left[\rho\, (dx \wedge dy \wedge dz)\right]\left(R_x \partial_x + R_y \partial_y + R_z \partial_z\right.$$

$W_{\vec{R}}(\vec{x}, \vec{y}) = \Omega(\vec{R}, \vec{x}, \vec{y})$

$$= \int_{\partial D} \rho \left(\cancel{dR_x}\, R_x\, dy\, dz + R_y\, dz \wedge dx + R_z\, dz \wedge dx\right) =$$

$$= \int d\left[\rho\, (R_x\, dy\, dz + R_y\, dz\, dx + R_z\, dz\, dx\right] =$$

$$= \int \left[\frac{\partial(\rho R_x)}{\partial x} + \frac{\partial(\rho R_y)}{\partial y} + \frac{\partial(\rho R_z)}{\partial z} \right] dx \wedge dy \wedge dz =$$

$$= \int \underbrace{\frac{1}{\rho}\left(\frac{\partial(\rho R^i)}{\partial x^i}\right)}\Omega$$

$$\operatorname{div} \vec{R} = \frac{1}{\rho} \frac{\partial(\rho R^i)}{\partial x^i}$$

There are other examples

$$\vec{F} = F_x \partial_x + F_y \partial_y + F_z \partial_z$$

$$\int_Z \vec{F} \cdot d\vec{l} = \int_Z \omega_F = \int (F_x\, dx + F_y\, dy + F_z\, dz) =$$

$$= \int_M d\,(F_1\, dx + F_2\, dy + F_3\, dz) =$$

$$= \int \Big(\overbrace{\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}}^{R_3}\Big) dx \wedge dy + \Big(\overbrace{\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}}^{R_1}\Big) dy \wedge dz +$$

$$+ \Big(\underbrace{\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}}_{R_2}\Big) dz \wedge dx =$$

$$\cancel{\int R_3\, dx \wedge dy + \int} \quad \int R_1\, dy \wedge dz + \int R_2\, dz \wedge dx + \int R_3\, dx \wedge dy$$

$$= \int_M \vec{R}\, d\vec{s}$$

$$\vec{R} = \operatorname{curl} \vec{F}$$

Exercise.

Prove $\oint_{\partial V} p\vec{n} \cdot d\vec{s} = \int \rho g$ ✓

(Archimed law).

Example
$(n = 3)$

$$\boxed{\text{functions}} \xrightarrow{\text{grad}} \boxed{\text{Vector field}} \xrightarrow{\text{curl}} \boxed{\text{Vector field}} \xrightarrow{\text{div}} \boxed{\text{function}}$$

$$\boxed{0\text{-forms}} \xrightarrow{d} \boxed{1\text{-forms}} \xrightarrow{d} \boxed{2\text{-forms}} \xrightarrow{d} \boxed{3\text{-forms}}$$

$\text{curl} \circ \text{grad} = 0.$
$\text{div} \circ \text{curl} = 0.$

$\text{curl}\, \vec{A} = 0 \Rightarrow A = \text{grad}\, U ?$
$\text{div}\, \vec{A} = 0 \Rightarrow \vec{A} = \text{curl}\, U$

$$d^2 = 0$$

$$d\omega = 0 \Rightarrow \omega = d\sigma ??$$

$$\dim \frac{\omega : d\omega = 0}{\omega : \omega = d\sigma} =$$

$=$ Betti number.

And many other things

Henrik.

24 X 2018