SECOND ORDER OPERATORS ON THE ALGEBRA OF DENSITIES AND A GROUPOID OF CONNECTIONS.

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Abstract. We consider the geometry of second order linear operators acting on the commutative algebra of densities on a (super)manifold introduced in our previous work. In the conventional language, operators on the algebra of densities correspond to operator pencils. This algebra has a natural invariant scalar product. We consider self-adjoint operators on the algebra of densities and analyze the corresponding "canonical operator pencils" passing through a given operator on densities of a particular weight. There are singular values for the pencil parameters. This leads to an interesting geometrical picture. In particular we obtain operators that depend on equivalence classes of connections and we study a groupoid of connections such that the orbits of this groupoid are these equivalence classes. Based on this point of view we analyze two examples: the second order canonical operator on an odd symplectic supermanifold appearing in the Batalin-Vilkovisky geometry and the Sturm-Liouville operator on the line related with classical constructions of projective geometry. We also consider a canonical second order semidensity arising on odd symplectic supermanifolds, which has some resemblance with mean curvature in Riemannian geometry.

1. Introduction

Second order linear operators appear in various problems in mathematical physics. A condition that an operator respects the geometrical structure of a problem under consideration usually fixes this operator almost uniquely or at least provides a great deal of information about it. For example the standard Laplacian $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ in Euclidean space $\mathbb{E}^3$ is defined uniquely (up to a constant) by the condition that it is invariant with respect to isometries of $\mathbb{E}^3$. Consider an arbitrary second order operator

$$\Delta=\frac{1}{2}\left(S^{ab}\partial_a\partial_b+T^a\partial_a+R\right) \tag{1}$$

acting on functions on manifold M. It defines on M symmetric contravariant tensor S^{ab} (its principal symbol). For example for a Riemannian manifold one can take a principal symbol $S^{ab}=g^{ab}$, where g^{ab} is the metric tensor (with upper indices). Then one can fix the scalar $R=0$ in (1) by the natural condition $\Delta 1=0$. What about the first order term $T^a\partial_a$ in the operator (1)? One can see that Riemannian structure fixes this term also. Indeed consider

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on M a divergence operator

$$\mathrm{div}_{\boldsymbol{\rho}}\mathbf{X} = \frac{1}{\rho(x)}\frac{\partial}{\partial x^a}\left(\rho(x)X^a\right)\,, \quad \text{where } \boldsymbol{\rho} = \rho(x)|D(x)| \text{ is an arbitrary volume form}\,, \tag{2}$$

and choose a volume form $\boldsymbol{\rho} = \boldsymbol{\rho}_g = \sqrt{\det g}|D(x)|$. On Riemannian manifold this volume form is defined uniquely (up to a constant factor) by covariance condition. Thus we come to second order operator Δ_g such that for an arbitrary function f

$$\Delta_g f = \frac{1}{2}\mathrm{div}_{\boldsymbol{\rho}_g}\,\mathrm{grad}\, f = \frac{1}{2}\frac{1}{\rho(x)}\frac{\partial}{\partial x^a}\left(\rho_g(x)g^{ab}\frac{\partial f(x)}{\partial x^b}\right) = \frac{1}{2}\left(\partial_a\left(g^{ab}\partial_b f\right) + \partial_a \log \rho_g(x) g^{ab}\partial_b f(x)\right) =$$

$$= \frac{1}{2}\left(g^{ab}\partial_a\partial_b f + \partial_a g^{ab}\partial_b f + \partial_a \log\sqrt{\det g} g^{ab}\partial_b f(x)\right)\,. \tag{3}$$

We see that Riemannian structure on manifold naturally defines a unique (up to a constant factor) second order operator on functions (the Laplace-Beltrami operator on Riemannian manifold M). For this operator the terms with first derivatives contain *a connection* ∇ *on volume forms.* This connection is defined by the condition that for an arbitrary volume form $\boldsymbol{\rho} = \rho(x)|D(x)|$, $\nabla_{\mathbf{X}}\boldsymbol{\rho} = \partial_{\mathbf{X}}\left(\frac{\boldsymbol{\rho}}{\boldsymbol{\rho}_g}\right)\boldsymbol{\rho}_g$:

$$\nabla_{\mathbf{X}}\boldsymbol{\rho} = X^a\left(\partial_a + \gamma_a\right)\rho(x)|D(x)| = X^a\partial_a\left(\frac{\boldsymbol{\rho}}{\boldsymbol{\rho}_g}\right)\boldsymbol{\rho}_g =$$

$$= X^a\partial_a\left(\frac{\rho(x)}{\sqrt{\det g}}\right)\sqrt{\det g}|D(x)| = X^a\left(\partial_a\rho(x) - \partial_a\log\left(\sqrt{\det g}\right)\right)|D(x)|\,. \tag{4}$$

(Here the symbol of connection $\gamma_a = -\partial_a(\rho_g(x)) = -\partial_a \log\sqrt{\det g}$.)

Consider another example: Let $S^{ab}(x)$ be an arbitrary symmetric tensor field (not necessarily non-degenerate) on manifold M equipped with affine structure with the connection on vector fields ∇: $\nabla_a\partial_b = \Gamma^c_{ab}\partial_c$. The affine structure defines the second order operator $S^{ab}\nabla_a\nabla_b = S^{ab}\partial_a\partial_b + \ldots$. Principal symbol of this operator is the tensor field $S^{ab}(x)$. The affine connection induces a connection ∇ on volume forms by the relation $\gamma_a = -\Gamma^b_{ab}$. In the case of a Riemannian manifold the tensor S^{ab} can be fixed by Riemannian metric, $S^{ab} = g^{ab}$, and the Levi-Civita Theorem provides a unique symmetric affine connection which preserves the Riemannian structure. Thus we arrive again to the Beltrami-Laplace operator (3).

Often it is important to consider differential operators on densities of arbitrary weight λ. For example a density of weight $\lambda = 0$ is an ordinary function, a volume form is a density of weight $\lambda = 1$. Wave function in Quantum Mechanics can be naturally considered as a half-density, i.e. a density of weight $\lambda = \frac{1}{2}$.

Analysis of differential operators on densities of arbitrary weights leads to beautiful geometric constructions. (See e.g. the works [7, 8, 16] and the book [18].) For example consider Diff(M)–modules which appear when we study operators on densities. Namely let $\mathcal{D}_\lambda(M)$ be a space of second order linear operators acting on densities of weight λ. This space has a natural structure of Diff (M)-module. In the work [8] Duval and Ovsienko classified these modules for all values of λ. In particular they wrote down explicit expressions for $\mathcal{D}_\lambda(M)$-isomorphisms $\varphi_{\lambda,\mu}$ between modules $\mathcal{D}_\lambda(M)$ and $\mathcal{D}_\mu(M)$ for $\lambda, \mu \neq 0, \frac{1}{2}, 1$. These isomorphisms have the following appearance: If an operator $\Delta_\lambda \in \mathcal{D}_\lambda(M)$ is given in local coordinates by the expression $\Delta_\lambda = A^{ij}(x)\partial_i\partial_j + A^i(x)\partial_i + A(x)$ then its image $\varphi_{\lambda,\mu}(\Delta_\lambda) = \Delta_\mu \in \mathcal{D}_\mu(M)$ is given in the same local coordinates by the expression $\Delta_\mu = B^{ij}(x)\partial_i\partial_j + B^i(x)\partial_i + B(x)$ where

$$\begin{cases} B^{ij} & = A^{ij}\,, \\ B^i & = \frac{2\mu-1}{2\lambda-1}A^i + \frac{2(\lambda-\mu)}{2\lambda-1}\partial_j A^{ji}\,, \\ B & = \frac{\mu(\mu-1)}{\lambda(\lambda-1)}A + \frac{\mu(\lambda-\mu)}{(2\lambda-1)(\lambda-1)}\left(\partial_j A^j - \partial_i\partial_j A^{ij}\right)\,. \end{cases} \tag{5}$$

At the exceptional cases $\lambda, \mu = 0, \frac{1}{2}, 1$, non-isomorphic modules occur.

In work [15] it was suggested the new approach to consider the commutative algebra $\mathcal{F}(M)$ of densities of all weights on a manifold M. This algebra possesses a canonical invariant scalar product. One can study differential operators on the algebra $\mathcal{F}(M)$. Due to the existence of the canonical scalar product it is possible to consider the notion of self-adjoint and antiself-adjoint differential operators on this algebra. Operators on the algebra $\mathcal{F}(M)$ can be identified with pencils of operators acting between the spaces of densities of various weights. Self-adjoint operators of second order on $\mathcal{F}(M)$ correspond to certain canonical pencils of operators with the same principal symbol and associated with connection on volume forms on M. This approach was suggested and developed in [15] for studying and classifying second order odd operators on odd symplectic manifolds (arising in the Batalin-Vilkovisky formalism) and on odd Poisson manifolds.

The canonical pencils of second order operators have the "universality" property: there is a unique such a pencil passing through an arbitrary second order operator acting on densities of arbitrary weight except for three singular cases. For example consider the operator Δ of weight 0 acting on densities of weight λ, $\Delta \in \mathcal{D}_\lambda(M)$, $\Delta = S^{ab}\partial_a\partial_b + \dots$, where S^{ab} is a symmetric contravariant tensor field. Then for an arbitrary weight λ except for the singular cases $\lambda = 0, \frac{1}{2}, 1$ there exists a canonical pencil of operators which passes through the operator Δ. These exceptional weights have deep geometrical and physical meaning. E.g. the failure to construct maps φ_λ, μ in equation (5) for singular cases is related with existence of non-equivalent $\mathrm{Diff}\,(M)$ modules (see for detail [8]). The space $\mathcal{D}_{1/2}(M)$ of operators on half-densities is drastically different from all other spaces $\mathcal{D}_\lambda(M)$, since for second order operators on half-densities there is a natural notion of a self-adjoint operator. This fact is of great importance for the Batalin-Vilkovisky geometry (see [12, 14]).

Applying the approach of the work [15] we study canonical pencils of second order operators of an arbitrary weight δ and analyze in detail the exceptional case when these operators act on densities of weight $\lambda = \frac{1-\delta}{2}$. (An operator has weight δ if it maps densities of weight λ into the densities of weight $\mu = \lambda + \delta$.) Such an operator pencil can be defined by a symmetric contravariant tensor density $\mathbf{S} = |D(x)|^\delta S^{ab}\partial_a \otimes \partial_b$ (this field defines the principal symbol) and a connection ∇ on volume forms. Specialising the pencil to the exceptional value of weight $\lambda = \frac{1-\delta}{2}$ we come to an operator which depends only on a equivalence class of connections. We assign to every field $\mathbf{S}$ the certain groupoid of connections $C_\mathbf{S}$. For the exceptional weight $\lambda = \frac{1-\delta}{2}$ the operator with the principal symbol $\mathbf{S}$ depends on an orbit of this groupoid.

This is particularly interesting in the case of odd symplectic structures. For symplectic structures (even or odd) there is no distinguished connection. On the other hand, if a symplectic structure is odd, then the Poisson tensor is symmetric and it defines the principal symbol $\mathbf{S}$ of an operator pencil of weight $\delta = 0$. It turns out that in spite of the absence of a distinguished connection, there exists a canonical class of connections on volume forms such that for them γ_a ($\nabla_a = \partial_a + \gamma_a$) vanishes in some Darboux coordinates. Connections of this canonical class belong to an orbit of the groupoid $C_{\mathbf{S}}$ and the corresponding operator on half-densities is the canonical operator introduced in [12]. This operator seems to be the correct clarification of the Batalin-Vilkovisky "odd Laplacian" [3]. (See for detail [14].) This approach may be used also in the case of Riemannian geometry where $\mathbf{S}$ is defined by a Riemannian metric. However in this case there exists a distinguished connection (Levi-Civita connection). We would like to mention article [2] where an interesting attempt to compare second order operators for even Riemannian and odd symplectic structures was made.

Another important case is a canonical pencil of operators of weight $\delta = 2$ on the line. By considering exceptional weights we come in particular to Schwarzian derivative.

The plan of the paper is as follows.

In the next section we consider second order operators on the algebra of functions. We come in this "naive" approach to preliminary relations between second order operators and connections on volume forms.

In the third section we consider first and second order operators on the algebra $\mathcal{F}(M)$ of all densities on a manifold M. First we define an invariant canonical scalar product on the algebra $\mathcal{F}(M)$. This algebra can be interpreted as a subalgebra of functions on an auxiliary manifold $\widehat{M}$. We consider derivations (which can be identified with vector fields on $\widehat{M}$) and first order operators on the algebra $\mathcal{F}(M)$. The canonical scalar product on $\mathcal{F}(M)$ leads to the canonical divergence of vector fields on $\widehat{M}$. We come in particular to the interpretation of Lie derivatives of densities as divergence-free vector fields on $\widehat{M}$.

After that we consider second order operators on the algebra $\mathcal{F}(M)$. We introduce our main construction: the self-adjoint second order operators on the algebra $\mathcal{F}(M)$, and consider the corresponding operator pencils. These considerations are due to the paper [15].

In the fourth section we consider operators of weight δ acting on densities of exceptional weight $\lambda = \frac{1-\delta}{2}$. For an arbitrary contravariant symmetric tensor density $\mathbf{S}$ of weight δ we consider groupoid $C_{\mathbf{S}}$. The orbits of the groupoid $C_{\mathbf{S}}$ are classes of connections such that operators with the principal symbol $\mathbf{S}$ acting on densities of the exceptional weight $\lambda = \frac{1-\delta}{2}$ depend only on these classes. This groupoid was first considered in [14, 15] for the Batalin-Vilkovisky geometry. We also give explicit description for corresponding Lie algebroids.

Then we consider various examples where these operators occur. We consider the example of operators of weight $\delta = 0$ acting on half-densities on a Riemannian manifold and on an odd symplectic supermanifold, and the example of operators of weight $\delta = 2$ acting on densities of weight $\lambda = -\frac{1}{2}$ on the line. In all these examples the operators depend on classes of connections on volume forms which vanish in special coordinates (such as Darboux coordinates for symplectic case and projective coordinates for the line).

Finally we consider the example of the canonical odd invariant half-density introduced in [13]. We show that this density depends on the class of affine connections which vanish in Darboux coordinates.

By differential operators throughout this text we mean only *linear* differential operators.

For standard material from supermathematics see [5], [17] and [20].

2. Second order operators on functions

In what follows M is a smooth manifold or supermanifold.

Let $L = T^a(x)\frac{\partial}{\partial x^a} + R(x)$ be a first order operator on functions on a manifold M. Under change of local coordinates $x^a = x^a(x^{a'})$ L transforms as follows:

$$L = T^a(x)\frac{\partial}{\partial x^a} + R(x) = T^a\left(x\left(x'\right)\right)x_a^{a'}\frac{\partial}{\partial x^{a'}} + R(x), \qquad \left(x_a^{a'} = \frac{\partial x^{a'}}{\partial x^a}\right).$$

We see that $T^a(x)\frac{\partial}{\partial x^a}$ is a vector field and $R(x)$ scalar field.

Now return to the second order operator (1) on a manifold M. Under a change of local coordinates $x^a = x^a(x^{a'})$

$$\Delta = \frac{1}{2}\left(S^{ab}(x)\partial_a\partial_b + T^a(x)\partial_a + R(x)\right) = \frac{1}{2}\underbrace{x_a^{a'}S^{ab}x_b^{b'}}_{S^{a'b'}}\partial_{a'}\partial_{b'} + \dots \tag{6}$$

Top component of operator Δ, $\frac{1}{2}S^{ab}\partial_a\otimes\partial_b$ defines symmetric contravariant tensor of rank 2 on M (the principal symbol of the operator $\Delta = \frac{1}{2}\left(S^{ab}(x)\partial_a\partial_b + \dots\right)$.

If tensor $S = 0$ then Δ becomes first order operator and $T^a\partial_a$ is a vector field. What about a geometrical meaning of the operator (6) in the case if principal symbol $S \neq 0$? To answer this question we introduce a scalar product $\langle\ ,\ \rangle$ in the space of functions on M and consider the difference of two second order operators $\Delta^+ - \Delta$, where Δ^+ is an operator adjoint to Δ with respect to the scalar product. A scalar product $\langle\ ,\ \rangle$ on the space of functions is defined by the following construction: an arbitrary volume form $\boldsymbol{\rho} = \rho(x)|D(x)|$ on M is chosen and

$$\langle f, g\rangle_{\boldsymbol{\rho}} = \int_M f(x)g(x)\rho(x)|D(x)|\,. \tag{7}$$

If x' are new local coordinates $x^a = x^a\left(x'\right)$ then in new coordinates the volume form $\boldsymbol{\rho}$ has appearance $\rho'(x')D(x') = \rho(x)|D(x)|$:

$$\boldsymbol{\rho} = \rho(x)|D(x)| = \rho(x(x'))\left|\frac{D(x)}{D(x')}\right|D(x') = \rho(x(x'))\det\left(\frac{\partial x^a}{\partial x^{a'}}\right)D(x') = \rho'(x')|D(x')|\,,$$

i.e.

$$\rho'(x') = \rho(x(x'))\det\left(\frac{\partial x^a}{\partial x^{a'}}\right).$$

In what follows we suppose that scalar product is well-defined: we suppose that M is compact orientable manifold and the oriented atlas of local coordinates is chosen (all local coordinates transformations have positive Jacobian: $\left|\frac{\partial x}{\partial x'}\right| = \det\left(\frac{\partial x}{\partial x'}\right) > 0$)) [1].

Now return to the operators Δ and the adjoint operator Δ^+. For operator Δ the operator Δ^+ is defined by relation $\langle\Delta f, g\rangle_{\boldsymbol{\rho}} = \langle f, \Delta^+ g\rangle_{\boldsymbol{\rho}}$. Integrating by parts we have

$$\langle\Delta f, g\rangle_{\boldsymbol{\rho}} = \int_M \underbrace{\frac{1}{2}\left(S^{ab}(x)\partial_a\partial_b f + T^a(x)\partial_a f + R(x)f\right)}_{\Delta f}g(x)\rho(x)|D(x)| =$$

[1] Coordinate volume form $|D(x)|$ is usually denoted by $dx^1dx^2\dots dx^n$. We prefer our notation $|D(x)|$ having in mind further considerations for supermanifolds.

$$\int_M f(x) \underbrace{\left(\frac{1}{2\rho}\partial_a\left(\partial_b\left(S^{ab}\rho g\right)\right) - \frac{1}{2\rho}\partial_a\left(T^a\rho g\right) + \frac{1}{2}Rg\right)}_{\Delta^+ g} \rho(x)|D(x)| = \langle f, \Delta^+ g\rangle_{\boldsymbol{\rho}}.$$

Principal symbols of operators Δ and Δ^+ coincide. Thus the difference $\Delta^+ - \Delta$ is *a first order operator*:

$$\Delta^+ - \Delta = \underbrace{\left(\partial_b S^{ab} - T^a + S^{ab}\partial_b \log\rho\right)\partial_a}_{\text{vector field}} + \text{scalar terms}\,. \tag{8}$$

Introducing the scalar product via chosen volume form $\boldsymbol{\rho}$ we come to the fact that for an operator $\Delta = \frac{1}{2}\left(S^{ab}\partial_a\partial_b + T^a\partial_a + R\right)$, and for an arbitrary volume form $\boldsymbol{\rho} = \rho(x)|D(x)|$ the expression $\left(\partial_b S^{ab} - T^a + S^{ab}\partial_b\log\rho\right)\partial_a$ is a vector field.

Claim : For an operator $\Delta = \frac{1}{2}\left(S^{ab}\partial_a\partial_b + T^a\partial_a + R\right)$ *the expression*

$$\gamma^a = \partial_b S^{ab} - T^a \tag{9}$$

is an upper connection on volume forms.

Before proving the claim we give two words about connections on the space of volume forms.

Connection ∇ on the space of volume forms defines the covariant derivative of volume forms with respect to vector fields. It obeys natural linearity properties and Leibnitz rule:

- $$\nabla_{\mathbf{X}}\left(\boldsymbol{\rho}_1 + \boldsymbol{\rho}_2\right) = \nabla_{\mathbf{X}}\left(\boldsymbol{\rho}_1\right) + \nabla_{\mathbf{X}}\left(\boldsymbol{\rho}_2\right)\,,$$
- for arbitrary functions f, g
  $$\nabla_{f\mathbf{X}+g\mathbf{Y}}\left(\boldsymbol{\rho}\right) = f\nabla_{\mathbf{X}}\left(\boldsymbol{\rho}\right) + g\nabla_{\mathbf{Y}}\left(\boldsymbol{\rho}\right)\,,$$
- and the Leibnitz rule:
  $$\nabla_{\mathbf{X}}\left(f(x)\boldsymbol{\rho}\right) = \partial_{\mathbf{X}} f(x)\left(\boldsymbol{\rho}\right) + f(x)\nabla_{\mathbf{X}}\left(\boldsymbol{\rho}\right)\,, \tag{10}$$

($\partial_{\mathbf{X}}$ is the directional derivative of functions along vector field $\mathbf{X}$: $\partial_{\mathbf{X}} f = X^a\frac{\partial f}{\partial x^a}$).

Denote by ∇_a covariant derivative with respect to vector field $\frac{\partial}{\partial x^a}$. Due to axioms (10)

$$\nabla_a(\rho(x)|D(x)|) = \left(\partial_a\rho(x) + \gamma_a\rho(x)\right)|D(x)|, \quad \text{where } \gamma_a|D(x)| = \nabla_a(|D(x)|)\,, \qquad \left(\partial_a = \frac{\partial}{\partial x^a}\right)\,.$$

Under changing of local coordinates $x^a = x^a(x^{a'})$ the symbol γ_a transforms in the following way:

$$\gamma_a|D(x)| = \nabla_a\left(|D(x)|\right) = x_a^{a'}\nabla_{a'}\left(\det\frac{\partial x}{\partial x'}|D(x')|\right) = x_a^{a'}\left(\partial_{a'}\left(\log\det\frac{\partial x}{\partial x'}\right) + \gamma_{a'}\right)|D(x)|\,, \tag{11}$$

i.e.

$$\gamma_a = x_a^{a'}\left(\gamma_{a'} + \partial_{a'}\log\det\left(\frac{\partial x}{\partial x'}\right)\right) = x_a^{a'}\gamma_{a'} - x_{b'}^b x_{ba}^{b'}\,.$$

(We use the standard formula that $\delta\log\det M = \mathrm{Tr}\,(M^{-1}\delta M)$. We use also short notations for derivatives: $x_a^{a'} = \frac{\partial x^{a'}(x)}{\partial x^a}$, $x_{bc}^{a'} = \frac{\partial x^{a'}}{\partial x^b\partial x^c}$. The summation over repeated indices is assumed.)

Let S^{ab} be a contravariant tensor field. One can assign to this tensor field an upper connection i.e. a contravariant derivative $\overset{S}{\nabla}$

$$\overset{S}{\nabla}{}^a\left(\rho|D(x)|\right) = \left(S^{ab}\partial_b + \gamma^a\right)\rho|D(x)|\,. \tag{12}$$

Remark 1. Given a contravariant tensor field $S^{ab}(x)$ an arbitrary connection ∇ (covariant derivative) induces the upper connection (contravariant derivative) $\overset{S}{\nabla}$: $\overset{S}{\nabla}{}^a = S^{ab}\nabla_b$. In the case where tensor field S^{ab} is non-degenerate, the converse implication is true also. A non-degenerate contravariant tensor field $S^{ab}(x)$ induces one-one correspondence between upper connections and usual connections. (Compare with the example 5 below where upper connection in general does not define the connection.)

Under changing of coordinates a symbol γ^a of upper connection (12) transforms in the following way:

$$\gamma^{a'} = x_a^{a'}\left(\gamma^a + S^{ab}\partial_b \log\det\left(\partial x'/\partial x\right)\right) . \tag{13}$$

Remark 2. From now on we refer to genuine connections (covariant derivatives) simply as connections. With some abuse of language we identify the connection ∇ with its symbol $\boldsymbol{\gamma} = \{\gamma_a\}$: $\gamma_a = \nabla_a|D(x)|$.

Remark 3. It is worth noting that the difference of two connections is a covector field, the difference of two upper connections is a vector field. In other words the space of all connections (upper connections) is affine space associated with linear space of covector (vector) fields.

Consider two important examples of connections on volume forms.

Example 1. An arbitrary volume form $\boldsymbol{\rho}$ defines a connection $\boldsymbol{\gamma}^{\boldsymbol{\rho}}$ due to the formula (4): $\boldsymbol{\gamma}^{\boldsymbol{\rho}}$: $\gamma_a^{\boldsymbol{\rho}} = -\partial_a \log\rho(x)$. This is a flat connection: its curvature vanishes: $F_{ab} = \partial_a\gamma_b - \partial_a\gamma_a = 0$. (Connection ∇ considered in the formula (4) is a flat connection defined by the volume form $\boldsymbol{\rho}_g$.)

Example 2. Let ∇ be affine connection on vector fields on manifold M. It defines connection on volume forms $\nabla \to -\mathrm{Tr}\,\nabla$ with $\gamma_a = -\Gamma^b_{ab}$ where Γ^a_{bc} are Christoffel symbols of affine connection.

It is easy to see that connection and upper connection define the covariant and respectively contravariant derivative of the densities of an arbitrary weight: for $\mathbf{s} = s(x)|D(x)|^\lambda \in \mathcal{F}_\lambda$

$$\nabla_a\mathbf{s} = \left(\partial_a s(x) + \lambda\gamma_a s(x)\right)|D(x)|^\lambda .$$

Respectively for upper connection

$$\nabla^a\mathbf{s} = \left(S^{ab}\partial_b s(x) + \lambda\gamma^a s(x)\right)|D(x)|^\lambda . \tag{14}$$

Sometimes we will use the concept of connection of the weight δ. This is a linear operation that transforms densities of weight λ to the densities of weight $\mu = \lambda + \delta$: for $\mathbf{s} = s(x)|D(x)|^\lambda \in \mathcal{F}_\lambda$

$$\nabla_a\mathbf{s} = \left(\partial_a s(x) + \lambda\gamma_a s(x)\right)|D(x)|^{\lambda+\delta},\ \nabla_a|D(x)| = \gamma_a|D(x)|^{\delta+1} .$$

Respectively for upper connection

$$\nabla^a\mathbf{s} = \left(S^{ab}\partial_b s(x) + \lambda\gamma^a s(x)\right)|D(x)|^{\lambda+\delta}, \quad \nabla^a|D(x)| = \gamma^a|D(x)|^{\delta+1}$$

Proof of the claim (9)*:* Consider a flat connection $\boldsymbol{\gamma}^{\boldsymbol{\rho}}$: $\gamma_a^{\boldsymbol{\rho}} = -\partial_a\log\rho$ defined by the volume form $\boldsymbol{\rho} = \rho(x)dx$ (see Example 1 above). Since the expression $\mathbf{Y} = \left(\partial_b S^{yab} - T^a + S^{ab}\partial_b\log\rho\right)\partial_a$ in (8) is a vector field (the principal symbol of the first order operator $\Delta^+ - \Delta$) and $S^{ab}\gamma_b^{\boldsymbol{\rho}} = -S^{ab}\partial_b\log\rho$ is an upper connection then the sum $Y^a + S^{ab}\gamma_b^{\boldsymbol{\rho}}$ is also upper connection:

$$S^{ab}\gamma_b^{\text{flat}} + Y^a = -S^{ab}\partial_b\log\rho + \left(\partial_b S^{ab} - T^a + S^{ab}\partial_b\log\rho\right)\partial_a = \partial_b S^{ab} - T^a .$$

Thus we have proved the claim.

Having in mind the result of the claim we can rewrite the operator Δ on functions in a more convenient form:

$$\Delta f = \frac{1}{2}\left(S^{ab}\partial_a\partial_b + T^a\partial_a + R\right)f = \frac{1}{2}\left(\partial_a S^{ab}\partial_b + L^a\partial_a + R\right)f, \qquad \text{with } L^a = T^a - \partial_b S^{ab}$$

We come to Proposition

Proposition 1. *For an arbitrary second order operator on functions on manifold M:*

$$\Delta = \frac{1}{2}\left(S^{ab}\partial_a\partial_b + T^a\partial_a + R\right) = \frac{1}{2}\left(\partial_a\left(S^{ab}\partial_b\dots\right) + L^a\partial_a + R\right)\,,$$

the principal symbol $\frac{1}{2}S^{ab}$ is symmetric contravariant tensor field of rank 2, the symbol $\gamma^a = -L^a = \partial_b S^{ba} - T^a$ defines an upper connection and the function $R = 2\Delta 1$ is a scalar:

$$\Delta f = \frac{1}{2}\partial_a\left(\underbrace{S^{ab}}_{tensor}\partial_b f\right) - \frac{1}{2}\underbrace{\gamma^a}_{connection}\partial_a f + \underbrace{\frac{1}{2}R}_{scalar}\, f.$$

Second order operators on functions are fully characterised by symmetric contravariant tensors of rank 2 (principal symbol), upper connections and scalar fields. In the case if principal symbol is non-degenerate ($\det S^{ab} \neq 0$ *upper connection defines a usual connection on volume forms: $\gamma_a = S^{-1}_{ab}\gamma^b$.*

3. Algebra of densities and second order operators on algebra of densities

3.1. **Algebra of densities $\mathcal{F}(M)$. Canonical scalar product.** We consider now the space of densities.

As usual we suppose that M is a compact orientable manifold with a chosen oriented atlas.

We say that $\mathbf{s} = s(x)|D(x)|^\lambda$ is a density of weight λ if under changing of local coordinates it is multiplied on the λ-th power of the Jacobian of the coordinate transformation:

$$\boldsymbol{s} = s(x)|D(x)|^\sigma = s\left(x\left(x'\right)\right)\left|\frac{Dx}{Dx'}\right|^\lambda |D(x')|^\lambda = s\left(x\left(x'\right)\right)\left(\det\left(\frac{Dx}{Dx'}\right)\right)^\lambda |D(x')|^\lambda\,.$$

(Density of weight $\lambda = 0$ is a usual function, density of weight $\lambda = 1$ is a volume form.)

Denote by $\mathcal{F}_\lambda = \mathcal{F}_\lambda(M)$ the space of densities of weight λ on the manifold M.

Denote by $\mathcal{F} = \mathcal{F}(M)$ the space of all densities on the manifold M.

The space $\mathcal{F}_\lambda$ of densities of the weight λ is a vector space. It is the module over the ring of functions on M. The space $\mathcal{F}$ of all densities is an algebra: If $\mathbf{s}_1 = s_1(x)|D(x)|^{\lambda_1} \in \mathcal{F}_{\lambda_1}$ and $\mathbf{s}_2 = s_2(x)|D(x)|^{\lambda_2} \in \mathcal{F}_{\lambda_2}$ then their product is the density $\mathbf{s}_1\cdot\mathbf{s}_2 = s_1(x)_2(x)Dx^{\lambda_1+\lambda_2} \in \mathcal{F}_{\lambda_1+\lambda_2}$.

On the algebra $\mathcal{F}(M)$ of all densities on M one can consider the canonical scalar product $\langle\ ,\ \rangle$ defined by the following formula: if $\mathbf{s}_1 = s_1(x)|D(x)|^{\lambda_1}$ and $\mathbf{s}_2 = s_2(x)|D(x)|^{\lambda_2}$ then

$$\langle\mathbf{s}_1,\mathbf{s}_2\rangle = \begin{cases} \int_M s_1(x)s_2(x)|D(x)|\,, & \text{if } \lambda_1+\lambda_2 = 1\,,\\ \\ 0 \text{ if} & \text{if } \lambda_1+\lambda_2 \neq 1\,.\end{cases} \tag{15}$$

(Compare this scalar product with a volume form depending scalar product $\langle\ ,\ \rangle_{\boldsymbol{\rho}}$ on algebra of functions introduced in formula (7).)

The canonical scalar product (15) was considered and intensively used in the work [15]. Briefly recall the constructions of this work.

Elements of the algebra $\mathcal{F}(M)$ are finite combinations of densities of different weights.

It is convenient to use a formal variable t in a place of coordinate volume form $|D(x)|$. An arbitrary density $\mathcal{F} \ni \mathbf{s} = s_1(x)|D(x)|^{\lambda_1} + \dots + s_k(x)|D(x)|^{\lambda_k}$ can be written as a function polynomial on a variable t:

$$\mathbf{s} = \mathbf{s}(x,t) = s_1(x)t^{\lambda_1} + \cdots + s_k(x)t^{\lambda_k}\,. \tag{16}$$

E.g. the density $s_1(x) + s_2(x)|D(x)|^{1/2} + s_3(x)|D(x)|$ can be rewritten as $s(x,t) = s_1(x) + \sqrt{t}s_2(x) + ts_3(x)$. In what follows we often will use this notation.

Remark 4. With some abuse of language we say that a function $f(x,t)$ is a function polynomial over t if it is a sum of finite number of monoms of arbitrary real degree over t, $f(x,t) = \sum_\lambda f_\lambda(x)t^\lambda$, $\lambda \in \mathbb{R}$.

What is a global meaning of the variable t? The relation (16) means that an arbitrary density on M can be identified with a polynomial function on the extended manifold $\widehat{M} = \det(TM)\backslash M$ which is the frame bundle of determinant bundle of M. The natural local coordinates on $\widehat{M}$ induced by local coordinates x^a on M are (x^a, t) where t is a coordinate which is in a place of volume form $|D(x)|$. Let x^a, $x^{a'}$ be two local coordinates on M. If (x^a, t) and $(x^{a'}, t')$ are local coordinates on $\widehat{M}$ induced by local coordinates x^a and $x^{a'}$ respectively then

$$x^{a'} = x^{a'}(x^a) \text{ and } t' = \det\left(\frac{\partial x'}{\partial x}\right) t\,. \tag{17}$$

If a function is a polynomial with respect to local variable t, then it is a polynomial with respect to local variable t' also. (As it was mentioned before we consider only oriented atlas, i.e. all changing of coordinates have positive determinant.)

It has to be emphasized that algebra $\mathcal{F}(M)$ of all densities on M can be identified with an algebra of functions on extended manifold $\widehat{M}$ *which are polynomial on* t. We *do not consider arbitrary functions on* t.

3.2. **Derivations of algebra of densities (=vector fields on the extended manifold).** Consider differential operators on the algebra $\mathcal{F}$. (We repeat that we consider only linear operators.)

Let $\mathbf{X}$ be a derivation of the algebra $\mathcal{F}$, Then for two arbitrary densities $\mathbf{s}_1, \mathbf{s}_2$

$$\mathbf{X}\left(\mathbf{s}_1 \cdot \mathbf{s}_2\right) = \left(\mathbf{X}\mathbf{s}_1\right) \cdot \mathbf{s}_2 + \mathbf{s}_1 \cdot \left(\mathbf{X}\mathbf{s}_2\right)\,, \qquad \text{(Leibnitz rule).}$$

The derivations of the algebra $\mathcal{F}(M)$ are vector fields on the extended manifold $\widehat{M}$, where coefficients are polynomials over t:

$$\mathbf{X} = X^a(x,t)\frac{\partial}{\partial x^a} + X^0(x,t)\widehat{\lambda} = \sum_\delta t^\delta \left(X^a_{(\delta)}(x)\frac{\partial}{\partial x^a} + X^0_{(\delta)}(x)\widehat{\lambda}\right)\,. \tag{18}$$

We introduced in this formula the Euler operator

$$\widehat{\lambda} = t\frac{\partial}{\partial t}$$

which is globally defined vector field on $\widehat{M}$ (see transformation law (17)). Euler operator $\widehat{\lambda}$ measures a weight λ of density: $\widehat{\lambda}\left(s(x)t^\lambda\right) = \lambda s(x)t^\lambda$.

There is a natural gradation in the space of vector fields. The vector field

$$\mathbf{X} = t^\delta\left(X^a(x)\partial_a + X^0(x)\widehat{\lambda}\right) \tag{19}$$

is the vector field of the weight δ. It transforms a density of weight λ to the density of weight $\lambda+\delta$.

Remark 5. From now on considering vector fields on extended manifold $\widehat{M}$ we suppose by default that coefficients of these vector fields are polynomial on t (see equation (18)).

Our next step is to consider adjoint operators with respect to canonical scalar product (15) on the algebra $\mathcal{F}$: operator $\hat{L}^+$ is adjoint to the operator L if for arbitrary densities $\mathbf{s}_1, \mathbf{s}_2$, $\langle \hat{L}\mathbf{s}_1, \mathbf{s}_2\rangle = \langle \mathbf{s}_1, \hat{L}^+\mathbf{s}_2\rangle$. One can see that

$$x^+ = x, \quad \left(\frac{\partial}{\partial x}\right)^+ = -\frac{\partial}{\partial x}, \text{ and } \quad \widehat{\lambda}^+ = 1-\widehat{\lambda}\,.$$

Check the last relation. Let $\mathbf{s}_1$ be a density of the weight λ_1 and $\mathbf{s}_2$ be a density of the weight λ_2. Then $\langle \widehat{\lambda}\mathbf{s}_1, \mathbf{s}_2\rangle = \lambda_1\langle \mathbf{s}_1, \mathbf{s}_2\rangle$ and $\langle \mathbf{s}_1, \widehat{\lambda}^+\mathbf{s}_2\rangle = \langle \mathbf{s}_1, (1-\widehat{\lambda})\mathbf{s}_2\rangle = (1-\lambda_2)\langle \mathbf{s}_1, \mathbf{s}_2\rangle$. In the case if $\lambda_1+\lambda_2=1$ these scalar products are equal since $\lambda_1 = 1-\lambda_2$. In the case if $\lambda_1+\lambda_2\neq 1$ these scalar products both vanish.

Example 3. Consider vector field on $\widehat{M}$ (derivation of algebra of densities): $\mathbf{X}$: $\mathbf{Xs} = \left(X^a(x,t)\partial_a + X^0(x,t)\widehat{\lambda}\right)\mathbf{s}(x,t)$. Then its adjoint operator is:

$$\mathbf{X}^+\colon\quad \mathbf{X}^+\mathbf{s} = \left[\left(X^a(x,t)\partial_a + X^0(x,t)\widehat{\lambda}\right)\right]^+\mathbf{s} = -\partial_a\left(X^a(x,t)\mathbf{s}\right)) + (1-\hat{\lambda})\left(X^0(x,t)\mathbf{s}\right)\,,$$

$$\mathbf{X}^+ = -\partial_a X^a(x,t) - X^a(x,t)\partial_a - X^0(x,t)\hat{\lambda} + \left(1-\widehat{\lambda}\right)X^0(x,t)\,.$$

Definition 1. (*Canonical divergence*). Divergence of vector field $\mathbf{X}$ on $\widehat{M}$ is defined by the formula

$$\operatorname{div}\mathbf{X} = -(\mathbf{X}+\mathbf{X}^+) = \partial_a X^a + \left(\widehat{\lambda}-1\right)X^0(x,t)\,. \tag{20}$$

In particular for vector field $\mathbf{X}$ of the weight δ, $\mathbf{X} = t^\delta\left(X^a\partial_a + X^0\widehat{\lambda}\right)$ (see equation(19))

$$\operatorname{div}\mathbf{X} = t^\delta\left(\partial_a X^a + (\delta-1)X^0\right)\,.$$

Divergence of vector field on $\mathcal{F}$ vanishes iff this vector field is anti-self-adjoint (with respect to canonical scalar product (15)): $\mathbf{X} = -\mathbf{X}^+ \Leftrightarrow \operatorname{div}\mathbf{X} = 0$.

Example 4. Divergence-less (=antiself-adjoint) vector fields of weight $\delta=0$ act on densities as Lie derivatives. Indeed consider vector field $\mathbf{X} = X^a\partial_a + X^0\widehat{\lambda}$ of the weight $\delta=0$. The condition $\operatorname{div}\mathbf{X} = \partial_a X^a - X^0 = 0$ means that $X^0 = \partial_a X^a$, i.e. $\mathbf{X} = X^a\partial_a + \partial_a X^a\widehat{\lambda}$. Hence for every λ, $\left.\mathbf{X}\right|_{\mathcal{F}_\lambda} = X^a\partial_a + \lambda\partial_a X^a$. This means that the action of divergence-free vector field $\mathbf{X}$ of weight $\delta=0$ on an arbitrary density is the Lie derivative of this density: for $\mathbf{s}\in\mathcal{F}_\lambda$

$$\mathbf{Xs} = (X^a\partial_a + \widehat{\lambda}\partial_a X^a)\mathbf{s} = \mathcal{L}_{\mathbf{X}}\mathbf{s} = \left(X^a\partial_a s(x) + \lambda\partial_a X^a s(x)\right)|D(x)|^\lambda\,. \tag{21}$$

If $\mathbf{X}$ is divergence-free vector field on $\widehat{M}$ of arbitrary weight then $\operatorname{div}\mathbf{X} = 0 \Leftrightarrow \mathbf{X} = t^\delta\left(X^a\partial_a + \partial_a X^a\frac{\widehat{\lambda}}{1-\delta}\right)$. We come to generalised Lie derivative: if $\delta\neq 1$ then for $\mathbf{s}\in\mathcal{F}_\lambda$

$$\mathcal{L}_{\mathbf{X}}\mathbf{s} = |D(x)|^\delta\left(X^a\partial_a + \partial_a X^a\frac{\widehat{\lambda}}{1-\delta}\right)\mathbf{s} = \left(X^a\partial_a s(x) + \frac{\lambda\partial_a X^a s}{1-\delta}\right)|D(x)|^{\lambda+\delta}\,. \tag{22}$$

One can consider a canonical projection p of vector fields on $\widehat{M}$ (derivation of algebra $\mathcal{F}(M)$) on vector densities on M. It is defined by the formula $p(\mathbf{X}) = \mathbf{X}\big|_{\mathcal{F}_0=C^\infty(M)}$. In coordinates p: $\mathbf{X} = X^a(x,t)\partial_a + X^0(x,t)\widehat{\lambda} \mapsto X^a(x,t)\partial_a$.

We say that vector field is vertical if $p\mathbf{X} = 0$, i.e. if $\mathbf{X} = X^0(x,t)\widehat{\lambda}$. Divergence of vertical vector field $\mathbf{X} = X^0(x,t)\widehat{\lambda}$ equals to $\operatorname{div}\mathbf{X} = (\widehat{\lambda} - 1)X^0(x,t)$.

Proposition 2. *Let* Π *be a projection of vector fields onto vertical vector fields such that* $\operatorname{div}\mathbf{X} = \operatorname{div}\,(\Pi\mathbf{X})$, *We have that*

$$\Pi\colon\quad \mathbf{X} = t^\delta\left(X^a\partial_a + X^0\widehat{\lambda}\right) \mapsto \Pi\mathbf{X} = t^\delta\left(\frac{\partial_a X^a}{\delta-1} + X^0\right)\widehat{\lambda}\,.$$

Every vector field $\mathbf{X}$ *of weight* $\delta \neq 1$ *can be uniquely decomposed as the sum of a vertical vector field and divergence-free vector field, generalised Lie derivative* (22)*) with respect to vector field* $p\mathbf{X}$*:*

$$\mathbf{X} = \Pi\mathbf{X} + (\mathbf{X} - \Pi\mathbf{X}) = \Pi\mathbf{X} + \mathcal{L}_{p\mathbf{X}}\,.$$

One can check the statements of this Proposition by straightforward applications of the formulae above.

What relations exist between the canonical divergence (20) of vector fields on extended manifold $\widehat{M}$ and a divergence of vector fields on a manifold M ? Let ∇ be an arbitrary connection on volume forms. It assigns to the vector field $\mathbf{X}$ on M a vector field $\mathbf{X}_{\boldsymbol{\gamma}}$ on the extended manifold $\widehat{M}$ by the formula $\mathbf{X}_{\boldsymbol{\gamma}} = X^a\left(\frac{\partial}{\partial x^a} + \widehat{\lambda}\gamma_a\right)$, where $\boldsymbol{\gamma} = \{\gamma_a\}$ is a symbol of connection ∇ in coordinates x, $(\nabla_a|D(x)| = \gamma_a|D(x)|)$. Connection ∇ defines the divergence of vector fields on M via the canonical divergence (20): for every vector field $\mathbf{X}$ on manifold M:

$$\operatorname{div}_{\boldsymbol{\gamma}}\mathbf{X} = \operatorname{div}\mathbf{X}_{\boldsymbol{\gamma}} = \left(\frac{\partial X^a}{\partial x^a} - \gamma_a X^a\right)\,. \tag{23}$$

A volume form $\boldsymbol{\rho} = \rho(x)|D(x)|$ defines flat connection $\gamma_a^{\boldsymbol{\rho}} = -\partial_a \log\rho$ (see equation (4) and example 1). The formula (23) implies the well-known formula (see also equation (2))) for divergence of vector field on manifold equipped with a volume form

$$\operatorname{div}_{\boldsymbol{\rho}}\mathbf{X} = \operatorname{div}\mathbf{X}_{\boldsymbol{\gamma}_{\boldsymbol{\rho}}} = \left(\frac{\partial X^a}{\partial x^a} + X^a\partial_a\log\rho\right) = \frac{1}{\rho}\frac{\partial}{\partial x^a}\left(\rho X^a\right)\,. \tag{24}$$

Considering a connection corresponding to an affine connection (see Example 2) we come to $\operatorname{div}_{\nabla}\mathbf{X} = \nabla_a X^a = (\partial_a X^a + X^a\Gamma^b_{ab})$. On Riemannian manifold M Riemannian metric defines connection on volume forms $\gamma_a = -\partial_a\log\sqrt{\det g}$ (via Levi-Civita connection or via invariant volume form $\boldsymbol{\rho}_g$). We come to

$$\operatorname{div}_g\mathbf{X} = \left(\frac{\partial X^a}{\partial x^a} + X^a\partial_a\log\sqrt{\det g}\right) = \frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x^a}\left(\sqrt{\det g}X^a\right)\,.$$

3.3. **Second order operators on the algebra $\mathcal{F}(M)$.** We will study now operators of order $\leqslant 2$ on the algebra $\mathcal{F}(M)$ of densities on manifold M.

First of all general remark about n-th order operators. 0-th order operator on the algebra $\mathcal{F}(M)$ is just multiplication operator on non-zero density. L is n-th order operator on the algebra $\mathcal{F}(M)$ $(n \geqslant 1)$ if for an arbitrary $\mathbf{s} \in \mathcal{F}(M)$ the commutator $[L, \mathbf{s}] = L \circ \mathbf{s} - \mathbf{s} \circ L$ is $(n-1)$-th order operator.

One can see that if L is an n-th order differential operator on $\mathcal{F}(M)$, then $L + (-1)^n L^+$ is n-th order operator and $L - (-1)^n L^+$ is an operator of the order $\leqslant n-1$. We have:

Proposition 3. *An arbitrary n-th order operator can be canonically decomposed on the sum of self adjoint and antiself-adjoint operators:*

$$L = \underbrace{\frac{1}{2}\left(L+(-1)^n L^+\right)}_{n\text{-th order operator}} + \underbrace{\frac{1}{2}\left(L-(-1)^n L^+\right)}_{\text{operator of the order } \leqslant n-1} .$$

An operator of the even order $n = 2k$ is a sum of self-adjoint operator of the order $2k$ and antiself-adjoint operator of the order $\leqslant 2k-1$, and an operator of the odd order $n = 2k+1$ is a sum of antiself-adjoint operator of the order $2k+1$ and self-adjoint operator of the order $\leqslant 2k$.

Operators of the order 0 are evidently self-adjoint.

Let $L = \mathbf{X} + B$ be first order antiself-adjoint operator, where $\mathbf{X}$ is a vector field on $\widehat{M}$ and B is a scalar term (density). We have $L + L^+ = 0 = \mathbf{X} + \mathbf{X}^+ + 2B = 0$. Hence $L = \mathbf{X} + \frac{1}{2}\operatorname{div}\mathbf{X}$.

Now study self-adjoint second order operators on $\mathcal{F}(M)$. Let Δ be *second order* operator of the weight δ on algebra $\mathcal{F}(M)$ of densities. In local coordinates

$$\Delta = \frac{t^\delta}{2}\left(\underbrace{S^{ab}(x)\partial_a\partial_b + \widehat{\lambda}B^a(x)\partial_a + \widehat{\lambda}^2 C(x)}_{\text{second order derivatives}} + \underbrace{D^a(x)\partial_a + \widehat{\lambda}E(x)}_{\text{first order derivatives}} + F(x)\right) . \tag{25}$$

Put normalisation condition

$$\Delta(1) = 0 \tag{26}$$

i.e. density $\frac{F|D(x)|^\delta}{2}$ in (25) vanishes ($F = 0$).

The operator Δ^+ adjoint to Δ equals to

$$\Delta^+ = \frac{1}{2}\left(\partial_b\partial_a\left(S^{ab}t^\delta\dots\right) - \partial_a\left(B^a\widehat{\lambda}^+ t^\delta\dots\right) + \left(C(\widehat{\lambda}^+)^2 t^\delta\dots\right) - \partial_a\left(D^a t^\delta\dots\right) + \left(E\widehat{\lambda}^+ t^\delta\dots\right)\right) =$$

$$\frac{t^\delta}{2}\left(S^{ab}\partial_a\partial_b + 2\partial_b S^{ba}\partial_a + \partial_a\partial_b S^{ba}\right) +$$

$$\frac{t^\delta}{2}\left(\left(\widehat{\lambda}+\delta-1\right)\left(B^a\partial_a + \partial_b B^b\right) + \left(\widehat{\lambda}+\delta-1\right)^2 C - \left(\widehat{\lambda}+\delta-1\right)E - D^a\partial_a - \partial_b D^b\right) .$$

Comparing this operator with operator (25) we see that the condition $\Delta^+ = \Delta$ implies that

$$\Delta = \frac{t^\delta}{2}\left(S^{ab}(x)\partial_a\partial_b + \partial_b S^{ba}\partial_a + \left(2\widehat{\lambda}+\delta-1\right)\gamma^a(x)\partial_a + \widehat{\lambda}\partial_a\gamma^a(x) + \widehat{\lambda}\left(\widehat{\lambda}+\delta-1\right)\theta(x)\right) . \tag{27}$$

Here for convenience we denote $\gamma^a = 2B^a$ and $\theta = C$. Studying how coefficients of the operator change under changing of coordinates we come to

Theorem 1. *(See* [15]*.) Let Δ be an arbitrary linear second order self-adjoint operator ($\Delta^+ = \Delta$) on the algebra $\mathcal{F}(M)$ of densities such that its weight equals δ and $\Delta(1) = 0$. Then in local coordinates this operator has the appearance* (27)*. The coefficients of this operator have the following geometrical meaning*

- $\mathbf{S} = t^\delta S^{ab}(x) = S^{ab}(x)|D(x)|^\delta$ *is symmetric contravariant tensor field-density of the weight* δ. *Under changing of local coordinates* $x^{a'} = x^{a'}(x^a)$ *it transforms in the following way:*
$$S^{a'b'} = J^{-\delta} x_a^{a'} x_b^{b'} S^{ab} \,,$$
- γ^a *is a symbol of upper connection-density of weight* δ *(see* (13) *above). Under changing of local coordinates* $x^{a'} = x^{a'}(x^a)$ *it transforms in the following way:*
$$\gamma^{a'} = J^{-\delta} x_a^{a'} \left(\gamma^a + S^{ab}\partial_b \log J\right) \,,$$
- *and* θ *transforms in the following way:*
$$\theta' = J^{-\delta}\left(\theta + 2\gamma^a \partial_a \log J + \partial_a \log J\, S^{ab}\partial_b \log J\right)$$

Here $J = \det\left(\frac{\partial x'}{\partial x}\right)$, *and* $x_a^{a'}$ *are short notations for derivatives:* $x_a^{a'} = \partial_a x^{a'}(x) = \frac{\partial x^{a'}(x)}{\partial x^a}$.

The object $\theta(x)|D(x)|^\delta$ is called Brans-Dicke function [2].

Remark 6. Let Δ be self-adjoint operator (27) and $\boldsymbol{\gamma}' = \{\gamma'_a\}$ be an arbitrary connection on volume forms ($\nabla|D(x)| = \gamma'_a|D(x)|$). Then for the upper connection-density in the equation (27) the difference $(\gamma^a - S^{ab}\gamma'_b)|D(x)|^\delta$ is a vector density of the weight δ. Respectively for Brans-Dicke function θ, the difference $\theta - \gamma'_a S^{ab}\gamma'_b$ is a density of the weight δ. (One can easy deduce this recalling the fact that the space of genuine connections as well as the space of upper connections is an affine space: if ∇ and ∇' are two different connections then their difference is (co)vector field: $\nabla' - \nabla = \gamma'_a - \gamma_a = X_a$.)

Corollary 1. *Given principal symbol* $\mathbf{S} = S^{ab}|D(x)|^\delta$ *of the weight* δ *and connection* $\boldsymbol{\gamma}$ *on volume forms canonically define the second order self-adjoint operator* (27) *with upper connection* $\gamma^a = S^{ab}\gamma_b$ *and Brans-Dicke function* $\theta(x) = \gamma_a S^{ab}\gamma_b$. *We denote this operator* $\Delta(\mathbf{S}, \boldsymbol{\gamma})$.

The inverse implication is valid in the case if $\mathbf{S} = S^{ab}|D(x)|^\delta$ *is non-degenerate: Second order self-adjoint operator* Δ *of weight* δ *which obeys normalisation condition* (26) *with non-degenerate principal symbol* $\mathbf{S}$ *uniquely defines connection* $\boldsymbol{\gamma}$ *such that* $\Delta = \Delta(\mathbf{S}, \boldsymbol{\gamma}) + \hat\lambda(\hat\lambda + \delta - 1)F$, *where* F *is a density of weight* δ, *and Brans-Dicke function* θ *equals to* $\theta = \gamma_a\gamma^a + F = \gamma_a S^{ab}\gamma_b + F$.

Consider examples

First consider the example of operator (27) with *degenerate* principal symbol $S^{ab}|D(x)|^\delta$.

Example 5. Let $\mathbf{X} = X^a \frac{\partial}{\partial x^a}$ and $\mathbf{Y} = Y^a \frac{\partial}{\partial x^a}$ be two vector fields on manifold M. Recall the operator of Lie derivative (see equation (21) $\mathcal{L}_\mathbf{X} = X^a\partial_a + \widehat\lambda\partial_a X^a$ and consider the operator
$$\Delta = \frac{1}{2}\left(\mathcal{L}_\mathbf{X}\mathcal{L}_\mathbf{Y} + \mathcal{L}_Y\mathcal{L}_\mathbf{X}\right) = \frac{1}{2}\left(X^a\partial_a + \widehat\lambda\partial_a X^a\right)\left(Y^a\partial_a + \widehat\lambda\partial_a Y^a\right) + (\mathbf{X} \leftrightarrow \mathbf{Y}) \,.$$
It is self-adjoint operator since Lie derivatvie is antiself-adjoint operator. Calculating this operator and comparing it with the expression (27) we come to
$$S^{ab} = X^aY^b + Y^bX^a, \gamma^a = \left(\partial_b X^b\right)Y^a + \left(\partial_b Y^b\right)X^a,\ \theta = \left(\partial_a X^a\right)\left(\partial_b Y^b\right) \,.$$

[2]Its transformation is similar to transformation of Brans-Dicke "scalar", in Kaluza-Klein reduction of 5-dimensional gravity to gravity+electromagnetism.

We see that in the general case (if dimension n of manifold is greater than 2) this operator has degenerate principal symbol and upper connection does not define uniquely genuine connection.

3.4. **Canonical pencil of operators.** Note that an operator L on the algebra $\mathcal{F}(M)$ of densities defines the pencil $\{L_\lambda\}$ of operators on spaces $\mathcal{F}_\lambda$: $L_\lambda = L\big|_{\mathcal{F}_\lambda}$. The self-adjoint operator Δ on the algebra of densities (see equation (27)) defines the canonical operator pencil $\{\Delta_\lambda\}, \lambda \in \mathbb{R}$, where

$$\Delta_\lambda = \Delta\big|_{\mathcal{F}_\lambda} =$$
$$= \frac{t^\delta}{2}\left(S^{ab}(x)\partial_a\partial_b + \partial_b S^{ba}\partial_a + (2\lambda+\delta-1)\,\gamma^a(x)\partial_a + \lambda\partial_a\gamma^a(x) + \lambda\,(\lambda+\delta-1)\,\theta(x)\right)\,. \quad (28)$$

It is canonical pencil defined by symmetric tensor density $\mathbf{S} = S^{ab}(x)|D(x)|^\delta$, upper connection γ^a and Brans-Dicke function $\theta(x)$. Respectively self-adjoint operator $\Delta(\mathbf{S}, \boldsymbol{\gamma})$ on the algebra of densities defined by tensor field-density $\mathbf{S} = S^{ab}(x)|D(x)|^\delta$ and genuine connection $\boldsymbol{\gamma}$ (see Corollary 1) defines the operator pencil $\Delta_\lambda(\mathbf{S}, \boldsymbol{\gamma})$ with Brans-Dicke function $\theta(x) = \gamma_a S^{ab}\gamma_b$.

Operator Δ_λ of the weight δ maps density of weight λ to densities of weight $\lambda+\delta$. Its adjoint operator Δ^+_λ maps density of weight $1-\delta-\lambda$ to densities of weight $1-\lambda$. The condition $\Delta = \Delta^+$ of self-adjointness of operator Δ is equivalent to the condition

$$\Delta^+_\lambda = \Delta_{1-\lambda-\delta}\,. \quad (29)$$

Example 6. Let $\boldsymbol{\rho} = \rho(x)|D(x)|$ be a volume form on the Riemannian manifold M. We can consider an operator Δ on functions such that $\Delta f = \mathrm{div}\,_{\boldsymbol{\rho}}\mathrm{grad}\, f = \frac{1}{2}\frac{1}{\rho(x)}\frac{\partial}{\partial x^a}\left(\rho(x)g^{ab}\frac{\partial f(x)}{\partial x^b}\right)$ (see the equations (2) and (24) (In the case if $\boldsymbol{\rho} = \sqrt{\det g}|D(x)|$ this is just the Laplace-Beltrami operator (3).) Using this operator consider the pencil

$$\Delta_\lambda = \boldsymbol{\rho}^\lambda \circ \Delta \circ \frac{1}{\boldsymbol{\rho}^\lambda}\colon \qquad \left(\text{for } \mathbf{s} \in \mathcal{F}_\lambda,\ \Delta_\lambda \mathbf{s} = \boldsymbol{\rho}^\lambda \mathrm{div}_{\boldsymbol{\rho}}\,\mathrm{grad}\left(\frac{\mathbf{s}}{\boldsymbol{\rho}^\lambda}\right)\right)\,.$$

One can see that this pencil corresponds to self-adjoint operator (see relation (29)). It coincides with the canonical pencil (28) of weight $\delta = 0$ in the case if principal symbol is defined by Riemannian metric $S^{ab} = g^{ab}$, connection is a flat connection defined by the volume form (see formula (4) and Example 1): $\gamma^a = -g^{ab}\partial_b \log\rho$, and $\theta = \gamma^a\gamma_a$.

The canonical pencil (28) has many interesting properties (see for detail [15]). In particular it has the following "universality" property:

Corollary 2. *Let Δ be an arbitrary (linear) second order operator of weight δ acting on the space $\mathcal{F}_\lambda$ of densities of weight λ, $\Delta\colon \mathcal{F}_\lambda \to \mathcal{F}_\mu$ $(\mu = \lambda+\delta)$. In the case if $\lambda \neq 0, \mu \neq 1$ and $\lambda + \mu \neq 1$ there exists a unique canonical pencil which passes through the operator Δ.*

If the operator Δ is given by the expression $\Delta = A^{ab}\partial_a\partial_b + A^a\partial_a + A(x)$ then the relations

$$\begin{cases} \frac{1}{2}S^{ab} & = A^{ab}\,, \\ \frac{1}{2}\left((2\lambda+\delta-1)\gamma^a + \partial_b S^{ba}\right) & = A^a\,, \\ \frac{1}{2}\left(\lambda\partial_a\gamma^a + \lambda(\lambda+\delta-1)\theta\right) & = A\,, \end{cases} \qquad (\lambda \neq 0, \lambda+\mu \neq 1, \mu \neq 1)\,.$$

uniquely define principal symbol, upper connection and Brans-Dicke field. Hence they uniquely define canonical pencil (28).

The "universality" property provides a beautiful interpretation of canonical map $\varphi_{\lambda,\mu}$ in the relation (5). Indeed due to this Corollary we "draw" the pencil through an arbitrary operator $\Delta_\lambda = A^{ij}(x)\partial_i\partial_j + A^i(x)\partial_i + A(x)$ acting on densities of weight λ. Then the image of this operator, operator $\Delta_\mu = \varphi_{\lambda,\mu}(\Delta_\lambda)$ is the operator of this pencil acting on densities of weight μ.

4. Operators depending on a class of connections

In this section we will return to second order differential operators on manifold M. We consider second order operators acting on densities of a specially chosen given weight.

4.1. **Operators of weight δ acting on densities of weight $\frac{1-\delta}{2}$.** The Corollary 2 states that for a second order operator $\Delta\colon \mathcal{F}_\lambda \to \mathcal{F}_\mu$ for all values of weights except the cases $\lambda = 0$, $\mu = 1$ or $\lambda+\mu = 1$ there is a unique canonical pencil (28) which passes through the operator Δ. Consider now an exceptional case when operator $\Delta\colon \mathcal{F}_\lambda \to \mathcal{F}_\mu$ is such that it has weight δ and $\lambda+\mu = 1$, i.e. it acts on densities of weight $\lambda = \frac{1-\delta}{2}$ and transforms them to densities of weight $\mu = \frac{1+\delta}{2}$.

Let Δ_{sing} be such an operator which belongs to the canonical pencil (28):

$$\Delta_{\text{sing}} = (\Delta_\lambda)\big|_{\lambda=\frac{1-\delta}{2}} = \frac{t^\delta}{2}\left(S^{ab}(x)\partial_a\partial_b + \partial_b S^{ba}\partial_a + \lambda\partial_a\gamma^a(x) + \lambda(\lambda+\delta-1)\theta(x)\right) =$$

$$= \frac{|D(x)|^\delta}{2}\left(S^{ab}(x)\partial_a\partial_b + \partial_b S^{ba}\partial_a + \frac{1-\delta}{2}\left(\partial_a\gamma^a(x) + \frac{\delta-1}{2}\theta(x)\right)\right). \tag{30}$$

On the other hand let Δ be an arbitrary second order differential operator of weight δ which acts on densities of weight $\frac{1-\delta}{2}$, $\Delta\colon \mathcal{F}_{\frac{1-\delta}{2}} \to \mathcal{F}_{\frac{1+\delta}{2}}$. Compare this operator with the operator Δ_{sing}. The operator Δ^+ which is adjoint to the operator Δ also acts from the space $\mathcal{F}_{\frac{1-\delta}{2}}$ into the space $\mathcal{F}_{\frac{1+\delta}{2}}$, since $\lambda+\mu = \frac{1-\delta}{2}+\frac{1+\delta}{2} = 1$ (compare with formula (29)). Hence an operator Δ can be canonically decomposed on the sum of second order self-adjoint operator and antiself-adjoint operator of the order $\leqslant 1$. Antiself-adjoint operator is just generalised Lie derivative (22):

$$\Delta^+ - \Delta = \mathcal{L}_{\mathbf{X}}\big|_{\mathcal{F}_{\frac{1-\delta}{2}}} = |D(x)|^\delta\left(X^a\partial_a + \frac{1}{2}\partial_a X^a\right).$$

Operator Δ_{sing} in formula (30) belongs to canonical pencil, it is self-adjoint operator: $\Delta^+_{\text{sing}} = \Delta_{\text{sing}}$. Difference of two self-adjoint operators of second order with the same principal symbol is self-adjoint operator of order $\leqslant 1$. Hence it is the zeroth order operator of multiplication on the density. These considerations imply the following statement:

Corollary 3. *Let Δ be an arbitrary second order operator of weight δ acting on the space of densities of weight $\frac{1-\delta}{2}$. Let $\mathbf{S} = S^{ab}|D(x)|^\delta$ be a principal symbol of the operator Δ.*

Let Δ_{sing} be an operator belonging to an arbitrary canonical pencil (28) *with the same weight δ and with the same principal symbol $\mathbf{S} = S^{ab}|D(x)|^\delta$.*

Then the difference $\Delta - \Delta_{\text{sing}}$ is an operator of order $\leqslant 1$. It equals to generalised Lie derivative (22) *with respect to a vector field +zeroth order operator of multiplication on a density:*

$$\Delta = \Delta_{\text{sing}} + \mathcal{L}_{\mathbf{X}} + F(x)|D(x)|^\delta.$$

In the case if operator Δ is self-adjoint, $\Delta^+ = \Delta$, then Lie derivative vanishes ($\mathbf{X} = 0$).

It follows from this Corollary that if the operator $\Delta\colon \mathcal{F}_{\frac{1-\delta}{2}} \to \mathcal{F}_{\frac{1+\delta}{2}}$ is self-adjoint operator then it is given in local coordinates by the expression

$$\Delta = \frac{1}{2}\left(S^{ab}(x)\partial_a\partial_b + \partial_b S^{ba}(x)\partial_a + U_{\mathbf{S}}(x)\right)|D(x)|^\delta\,,$$

where

$$U_{\mathbf{S}}(x)|D(x)|^\delta = \frac{1-\delta}{2}\left(\partial_a\gamma^a(x) + \frac{\delta-1}{2}\theta(x)\right)|D(x)|^\delta + F(x)|D(x)|^\delta\,.$$

Here γ^a, θ are upper connection and Brans-Dicke field defining the pencil (28), and $F(x)|D(x)|^\delta$ is a density. In particular the self-adjoint operator $\Delta\colon \mathcal{F}_{\frac{1-\delta}{2}} \to \mathcal{F}_{\frac{1+\delta}{2}}$ belongs to the canonical pencil defined by the same principal symbol $\mathbf{S}$, and upper connection γ^a but different $\theta' = \theta - \frac{4F}{(\delta-1)^2}$. It may belong to many other pencils with different upper connections. Self-adjoint operator Δ acting on densities of the exceptional weight $\lambda = \frac{1-\delta}{2}$ *does not define uniquely the canonical pencil.* Thus we come to

4.2. **Groupoid of connections.** We define now a groupoid $C_{\mathbf{S}}$ of connections associated with contravariant tensor field-density $\mathbf{S} = S^{ab}|D(x)|^\delta$ of weight δ.

Consider a space $\mathbf{A}$ of all connections on volume forms (covariant derivatives of volume forms) on manifold M. This is an affine space associated to the vector space of covector fields on M: difference of two connections ∇ and ∇' is covector field (differential 1-form):

$$\nabla - \nabla' = \boldsymbol{\gamma} - \boldsymbol{\gamma}' = \mathbf{X} = X_a dx^a\,, \text{ where } X_a = \gamma_a - \gamma'_a\,.$$

Define a set of arrows as a set $\{\boldsymbol{\gamma}\xrightarrow{\mathbf{X}}\boldsymbol{\gamma}'\}$ such that $\boldsymbol{\gamma}, \boldsymbol{\gamma}' \in \mathbf{A}$ and $\boldsymbol{\gamma}' = \boldsymbol{\gamma} + \mathbf{X}$, where $\mathbf{X}$, difference of connections is a covector field. We come to trivial groupoid of connections:

$$-\left(\boldsymbol{\gamma}\xrightarrow{\mathbf{X}}\boldsymbol{\gamma}'\right) = \boldsymbol{\gamma}'\xrightarrow{-\mathbf{X}}\boldsymbol{\gamma}\,, \qquad \left(\boldsymbol{\gamma}_1\xrightarrow{\mathbf{X}}\boldsymbol{\gamma}_2\right) + \left(\boldsymbol{\gamma}_2\xrightarrow{\mathbf{Y}}\boldsymbol{\gamma}_3\right) = \gamma_1\xrightarrow{\mathbf{X}+\mathbf{Y}}\gamma_3\,. \tag{31}$$

Pick an arbitrary contravariant symmetric tensor field-density of the weight δ: $\mathbf{S}(x) = S^{ab}(x)|D(x)|^\delta$. The tensor field-density $\mathbf{S}$ and an arbitrary connection $\boldsymbol{\gamma}$ define the self-adjoint operator $\Delta(\mathbf{S},\boldsymbol{\gamma})$ on the algebra of densities. It is the operator defined in the equation (27) with principal symbol $\mathbf{S}$, with upper connection $\gamma^a = S^{ab}\gamma_b$ and Brans-Dicke function $\theta = \gamma_a\gamma^a$ (see the Corollary 1). Consider corresponding pencil of operators and the operator $\Delta_{\text{sing}}(\mathbf{S},\boldsymbol{\gamma})$ which belongs to this pencil and acts on densities of weight $\frac{1-\delta}{2}$:

$$\Delta_{\text{sing}}(\mathbf{S},\boldsymbol{\gamma}) = \Delta(\mathbf{S},\boldsymbol{\gamma})\big|_{\mathcal{F}_{\frac{1-\delta}{2}}} =$$

$$= \frac{|D(x)|^\delta}{2}\left(S^{ab}(x)\partial_a\partial_b + \partial_b S^{ba}\partial_a + \frac{1-\delta}{2}\left(\partial_a\gamma^a(x) + \frac{\delta-1}{2}\gamma_a\gamma^a(x)\right)\right)\,. \tag{32}$$

Thus an arbitrary contravariant symmetric tensor field-density $\mathbf{S} = S^{ab}(x)|D(x)|^\delta$ of the weight δ and an arbitrary connection $\boldsymbol{\gamma}$ defines self-adjoint operator $\Delta_{\text{sing}}(\mathbf{S},\boldsymbol{\gamma})$ by the relation (32). "Pseudoscalar" part of this operator is equal to

$$U_{\mathbf{S},\boldsymbol{\gamma}}(x)|D(x)|^\delta = \left(\frac{1-\delta}{2}\right)\left(\partial_a\gamma^a(x) + \frac{\delta-1}{2}\gamma_a\gamma^a(x)\right)\frac{|D(x)|^\delta}{2}\,. \tag{33}$$

Let $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}'$ be two different connections. Difference of two operators $\Delta_{\text{sing}}(\mathbf{S},\boldsymbol{\gamma})$ and $\Delta_{\text{sing}}(\mathbf{S},\boldsymbol{\gamma}')$ with the same principal symbol $\mathbf{S} = S^{ab}(x)|D(x)|^{\delta}$ is the scalar density of the weight δ. Calculate this density. If $\boldsymbol{\gamma}' = \boldsymbol{\gamma} + \mathbf{X}$ then

$$\Delta_{\text{sing}}(\mathbf{S},\boldsymbol{\gamma}') - \Delta_{\text{sing}}(\mathbf{S},\boldsymbol{\gamma}) = U_{\mathbf{S},\boldsymbol{\gamma}'}(x)|D(x)|^{\delta} - U_{\mathbf{S},\boldsymbol{\gamma}}(x)|D(x)|^{\delta} =$$

$$\left(\frac{1-\delta}{4}\right)\left(\partial_a\gamma'^a(x) + \frac{\delta-1}{2}\gamma'_a\gamma'^a(x) - \partial_a\gamma^a(x) - \frac{\delta-1}{2}\gamma_a\gamma^a(x)\right)|D(x)|^{\delta} =$$

$$\left(\frac{1-\delta}{4}\right)\left(\partial_a(S^{ab}X_b) + (\delta-1)\gamma_a(S^{ab}X_b) + \frac{\delta-1}{2}X_aS^{ab}X_b\right)|D(x)|^{\delta} =$$

$$\frac{1-\delta}{4}\left(\operatorname{div}_{\boldsymbol{\gamma}}\mathbf{X} + \frac{\delta-1}{2}\mathbf{X}^2\right). \tag{34}$$

Here $\operatorname{div}_{\boldsymbol{\gamma}}\mathbf{X}$ is the divergence of vector density $\mathbf{X}$ on M with respect to connection $\boldsymbol{\gamma}$ (see (23)). With some abuse of notation we denote the covector field $X_a dx^a$ and vector density of the weight δ, $X^a|D(x)|^{\delta} = S^{ab}X_b|D(x)|^{\delta}$ by the same letter $\mathbf{X}$.

Definition 2. Let $\mathbf{S} = S^{ab}(x)|D(x)|^{\delta}$ be contravariant symmetric tensor field-density of the weight δ. The groupoid $C_{\mathbf{S}}$ is a subgroupoid of arrows $\boldsymbol{\gamma} \xrightarrow{\mathbf{X}} \boldsymbol{\gamma}'$ of trivial groupoid (31) such that the operators $\Delta_{\text{sing}}(\mathbf{S},\boldsymbol{\gamma})$ and $\Delta_{\text{sing}}(\mathbf{S},\boldsymbol{\gamma}')$ defined by the formula (32) coincide:

$$C_{\mathbf{S}} = \{\text{Groupoid of arrows } \boldsymbol{\gamma} \xrightarrow{\mathbf{X}} \boldsymbol{\gamma}' \text{such that } \Delta_{\text{sing}}(\mathbf{S},\boldsymbol{\gamma}') = \Delta_{\text{sing}}(\mathbf{S},\boldsymbol{\gamma})\}. \tag{35}$$

Using the formula (34) for difference of operators $\Delta_{\text{sing}}(\mathbf{S},\boldsymbol{\gamma}')$ and $\Delta_{\text{sing}}(\mathbf{S},\boldsymbol{\gamma})$ rewrite the definition (35) of groupoid $C_{\mathbf{S}}$ in the following way:

$$C_{\mathbf{S}} = \{\text{Groupoid of arrows } \boldsymbol{\gamma} \xrightarrow{\mathbf{X}} \boldsymbol{\gamma}' \text{ such that } \operatorname{div}_{\boldsymbol{\gamma}}\mathbf{X} + \frac{\delta-1}{2}\mathbf{X}^2 = 0\}.$$

In other words the arrow $\boldsymbol{\gamma} \xrightarrow{\mathbf{X}} \boldsymbol{\gamma}'$ belongs to the groupoid $C_{\mathbf{S}}$ if two canonical pencils $\Delta_\lambda(\mathbf{S},\boldsymbol{\gamma})$ and $\Delta_\lambda(\mathbf{S},\boldsymbol{\gamma}')$ intersect at the operator $\Delta_{\text{sing}}(\mathbf{S},\boldsymbol{\gamma})$.

We consider the case $\delta \neq 1$, The case $\delta = 1$ is trivial [3].

Denote by $[\boldsymbol{\gamma}]$ the orbit of a connection $\boldsymbol{\gamma}$ in the groupoid $C_{\mathbf{S}}$

$$[\gamma] = \{\boldsymbol{\gamma}' : \quad \gamma \xrightarrow{\mathbf{X}} \boldsymbol{\gamma}' \in C_{\mathbf{S}}\}.$$

Proposition 4. *An arbitrary contravariant symmetric tensor field $\mathbf{S} = S^{ab}(x)|D(x)|^{\delta}$ of the weight δ defines the groupoid of connections $C_{\mathbf{S}}$ and the family of second order differential operators of the order δ and acting on densities of the weight $\frac{1-\delta}{2}$:*

$$\Delta([\boldsymbol{\gamma}]) = \Delta_{\text{sing}}(\mathbf{S},\boldsymbol{\gamma}) \colon \mathcal{F}_{\frac{1-\delta}{2}} \to \mathcal{F}_{\frac{1+\delta}{2}}.$$

Operators of this family have the same principal symbol and they depend on equivalence classes of connections which are orbits in the groupoid $C_{\mathbf{S}}$.

[3]In this case all the operators $\Delta_{\text{sing}}(\mathbf{S},\boldsymbol{\gamma})$ do not depend on connection $\boldsymbol{\gamma}$. Principal symbol $\mathbf{S} = S^{ab}|D(x)|$ defines the canonical operator $\Delta(\mathbf{S}) \colon \mathcal{F}_0 \to \mathcal{F}_1$ such that in local coordinates $\Delta(\mathbf{s})f = \partial_a\left(S^{ab}\partial_b f\right)|D(x)|$. The groupoid $C_{\mathbf{S}}$ is the trivial groupoid of all connections.

Remark 7. Let $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2$ and $\boldsymbol{\gamma}_3$ be three arbitrary connections. Consider corresponding arrows $\boldsymbol{\gamma}_1 \xrightarrow{\mathbf{X}} \boldsymbol{\gamma}_2$, $\boldsymbol{\gamma}_2 \xrightarrow{\mathbf{Y}} \boldsymbol{\gamma}_3$ and $\boldsymbol{\gamma}_1 \xrightarrow{\mathbf{X}+\mathbf{Y}} \boldsymbol{\gamma}_3$. We have that $\boldsymbol{\gamma}_1 \xrightarrow{\mathbf{X}} \boldsymbol{\gamma}_2 + \boldsymbol{\gamma}_2 \xrightarrow{\mathbf{Y}} \boldsymbol{\gamma}_3 = \boldsymbol{\gamma}_1 \xrightarrow{\mathbf{X}+\mathbf{Y}} \boldsymbol{\gamma}_3$. This means that for non-linear differential equation $\operatorname{div}_{\boldsymbol{\gamma}} \mathbf{X} + \frac{\delta-1}{2}\mathbf{X}^2 = 0$ the following property holds:

$$\begin{cases} \operatorname{div}_{\boldsymbol{\gamma}_1} \mathbf{X} + \frac{\delta-1}{2}\mathbf{X}^2 = 0 \\ \operatorname{div}_{\boldsymbol{\gamma}_2} \mathbf{Y} + \frac{\delta-1}{2}\mathbf{Y}^2 = 0 \end{cases} \Rightarrow \operatorname{div}_{\boldsymbol{\gamma}_1} (\mathbf{X}+\mathbf{Y}) + \frac{\delta-1}{2}(\mathbf{X}+\mathbf{Y})^2 = 0\,.$$

It follows from (34) the cocycle condition that the sum of left-hand side of first two equations is equal to the left hand-sight of the third equation.

Remark 8. Let $\boldsymbol{\rho}$ be an arbitrary volume form. Using the operator $\Delta([\boldsymbol{\gamma}]) = \Delta_{\text{sing}}(\mathbf{S}, \boldsymbol{\gamma})$ one can consider second order operator

$$\Delta\colon \Delta f = \boldsymbol{\rho}^{-\frac{1+\delta}{2}} \Delta\left([\boldsymbol{\gamma}]\right) \left(\boldsymbol{\rho}^{\frac{1-\delta}{2}} f(x)\right)$$

on functions depending on volume form $\boldsymbol{\rho}$. Calculating one comes to

$$\Delta f = \frac{1}{2}\left(S^{ab}\partial_a\partial_b + \partial_b S^{ba}\partial_a + (\delta-1)\gamma^a_{\boldsymbol{\rho}}\partial_a + \frac{1}{t^\delta}\left(U_{\mathbf{S},\boldsymbol{\gamma}} - U_{\mathbf{S},\boldsymbol{\gamma}^{\boldsymbol{\rho}}}\right)\right) f$$

Here $\boldsymbol{\gamma}^{\boldsymbol{\rho}}\colon\ \gamma_a = -\partial_a \log\rho$ is a flat connection defined by the volume form $\boldsymbol{\rho} = \rho(x)|D(x)|$, $\gamma^a = S^{ab}\gamma_b$ and $U_{\mathbf{S},\boldsymbol{\gamma}}/t^\delta$ is a "pseudoscalar" part (33) of the operator (32). The difference $U_{\mathbf{S},\boldsymbol{\gamma}} - U_{\mathbf{S},\boldsymbol{\gamma}_{\boldsymbol{\rho}}}$ is a density of weight δ (see Corollary 3 and equation (34)).

We consider now examples of groupoids and corresponding operators $\Delta_{\text{sing}}(\mathbf{S}, \boldsymbol{\gamma})$.

4.3. **Groupoid $C_{\mathbf{S}}$ for a Riemannian manifold.** Let M be Riemannian manifold equipped with Riemannian metric G. (As always we suppose that M is orientable compact manifold with a chosen oriented atlas). Riemannian metric defines principal symbol $\mathbf{S} = G^{-1}$. In local coordinates $S^{ab} = g^{ab}$ ($G = g_{ab}dx^a dx^b$). It is principal symbol of operator of weight $\delta = 0$.

Let $\boldsymbol{\gamma}$ be an arbitrary connection on volume forms. The differential operator $\Delta = \Delta_{\text{sing}}(G^{-1}, \boldsymbol{\gamma})$ with weight $\delta = 0$ transforms half-densities to half-densities. Due to the formulae (32), (33) this operator equals to

$$\Delta_{\text{sing}}(G^{-1}, \boldsymbol{\gamma})\colon \mathcal{F}_{\frac{1}{2}} \to \mathcal{F}_{\frac{1}{2}}, \quad \Delta_{\text{sing}}(G^{-1}, \boldsymbol{\gamma}) = \frac{1}{2}\left(g^{ab}\partial_a\partial_b + \partial_b g^{ba}\partial_a + \frac{1}{2}\partial_a\gamma^a - \frac{1}{4}\gamma_a\gamma^a\right).$$

We come to the groupoid

$$C_G = \left\{\text{Groupoid of arrows } \boldsymbol{\gamma} \xrightarrow{\mathbf{X}} \boldsymbol{\gamma}' \text{ such that } \operatorname{div}_{\boldsymbol{\gamma}}\mathbf{X} - \frac{1}{2}\mathbf{X}^2 = 0\right\}$$

and to the operator on half-densities depending on the class of connections

$$\Delta\left([\boldsymbol{\gamma}]\right) = \frac{1}{2}\left(g^{ab}\partial_a\partial_b + \partial_b g^{ba}\partial_a + \frac{1}{2}\partial_a\gamma^a - \frac{1}{4}\gamma_a\gamma^a\right).$$

On Riemannian manifold one can consider distinguished Levi-Civita connection on vector fields. This connection defines the connection $\boldsymbol{\gamma}^G$ on volume forms, such that $\gamma^G_a = -\Gamma^b_{ab} = -\partial_a \log\sqrt{\det g}$, where Γ^a_{bc} are Christoffel symbols of Levi-Civita connection. (We also call this connection on volume forms, Levi-Civita connection.) Consider the orbit, equivalence classes

$[\boldsymbol{\gamma}^G]$ in the groupoid C_G of Levi-Civita connection $\boldsymbol{\gamma}^G$. This orbit defines the distinguished operator on half-densities on Riemannian manifold:

$$\Delta = \Delta_G\left([\boldsymbol{\gamma}^G]\right) .$$

One can always choose *special* local coordinates (x^a) such that in these coordinates $\det g = 1$. In these local coordinates $\gamma_a^G = 0$ and the distinguished operator Δ on half-densities has the appearance:

$$\Delta = \frac{1}{2}\left(g^{ab}\partial_a\partial_b + \partial_b g^{ba}\partial_a\right), \quad \text{for}\, \mathbf{s} = s(x)|D(x)|^{\frac{1}{2}},\ \Delta\mathbf{s} = \frac{1}{2}\left(\partial_b\left(g^{ba}\partial_a s(x)\right)\right)|D(x)|^{\frac{1}{2}} .$$

The differential equation

$$\operatorname{div}_{\boldsymbol{\gamma}}\mathbf{X} - \frac{1}{2}\mathbf{X}^2 = 0$$

defining groupoid C_G has the following appearance in these coordinates:

$$\frac{\partial X^a(x)}{\partial x^a} - \frac{1}{2}X^a(x)X_a(x) = 0 .$$

All connections $\boldsymbol{\gamma}$ such that they have appearance $\gamma_a(x) = X_a(x)$ in these special coordinates, where $X_a(x)$ is a solution of this differential equation, belong to the orbit $[\boldsymbol{\gamma}^G]$.

The operator $\Delta\left([\boldsymbol{\gamma}^G]\right)$ belongs in particular to the canonical pencil associated with the Beltrami-Laplace operator (see the example 6).

On the other hand let $\boldsymbol{\gamma}$ be an arbitrary connection and let $\boldsymbol{\rho}$ be an arbitrary volume form on the Riemannian manifold M. One can assign to volume form $\boldsymbol{\rho}$ the flat connection $\boldsymbol{\gamma}^{\boldsymbol{\rho}}$: $\gamma_a^{\boldsymbol{\rho}} = -\partial_a \log\rho$. Consider operator $\frac{1}{\sqrt{\boldsymbol{\rho}}}\Delta\left([\boldsymbol{\gamma}]\right)\sqrt{\boldsymbol{\rho}}$ on functions (see Remark 8.) We come to scalar operator on functions

$$\Delta f = \frac{1}{2}\left(\partial_a\left(g^{ab}\partial_b f\right) - \gamma^{\boldsymbol{\rho} a}\partial_a + R\right)$$

where scalar function R equals to

$$R = U_{G,\boldsymbol{\gamma}} - U_{G,\boldsymbol{\gamma}^{\boldsymbol{\rho}}} = \frac{1}{2}\operatorname{div}\mathbf{X} - \frac{1}{4}\mathbf{X}^2 .$$

Here vector field $\mathbf{X}$ is defined by the difference of the connections: $\mathbf{X} = \boldsymbol{\gamma} - \boldsymbol{\gamma}^{\boldsymbol{\rho}}$.

It is interesting to compare formulae of this subsection with constructions in the paper [2] for a case of Riemannian structure.

Our next example is a groupoid on odd symplectic supermanifold. Before discussing it sketch shortly what happens if we consider supermanifolds instead manifolds.

4.4. **Supermanifold case.** Let M be $n|m$-dimensional supermanifold. Denote local coordinates of supermanifold by $z^A = (x^a, \theta^\alpha)$ $(a = 1, \dots, n; \alpha = 1, \dots, m)$. Here x^a are even coordinates and θ^α odd coordinates: $z^A z^B = (-1)^{p(A)p(B)} z^B z^A$, where $p(z^A)$, or shortly $p(A)$ is a parity of coordinate z^A; $(p(x^a) = 0, p(\theta^\alpha) = 1)$.

We would like to study second order linear differential operators $\Delta = S^{AB}\partial_A\partial_B + \dots$. Principal symbol of this operator is supersymmetric contravariant tensor field $\mathbf{S} = S^{AB}$. This field may be even or odd:

$$S^{AB} = (-1)^{p(A)p(B)} S^{BA},\ p(S^{AB}) = p(\mathbf{S}) + p(A) + p(B) .$$

The analysis of second order operators can be performed in supercase in a way similar to usual case. We have just to worry about sign rules. E.g. the formula (28) for canonical pencil of operators has to be rewritten in the following way

$$\Delta_\lambda = \frac{t^\delta}{2}\left(S^{AB}(x)\partial_B\partial_A + (-1)^{p(A)p(\mathbf{S}+1)}\partial_B S^{BA}\partial_A\right) +$$

$$+\frac{t^\delta}{2}\left((2\lambda+\delta-1)\,\gamma^A(x)\partial_A + (-1)^{p(A)p(\mathbf{S}+1)}\lambda\partial_A\gamma^A(x) + \lambda\,(\lambda+\delta-1)\,\theta(x)\right)\,. \qquad (36)$$

Here Δ is even (odd) operator if principal symbol $\mathbf{S}$ is even (odd) tensor field (see for detail [15]).

In the case if $\mathbf{S}$ is an even tensor field and it is non-degenerate then it defines Riemannian structure on (super)manifold M. We come to groupoid $C_{\mathbf{S}}$ in a same way as in a case of usual Riemannian manifold considered in the previous subsection. (We just must worry about signs arising in calculations.) In particular for even Riemannian supermanifold there exists distinguished Levi-Civita connection which canonically induces the unique connection on volume forms. This connection is a flat connection of the canonical volume form:

$$\boldsymbol{\rho}_g = \sqrt{\operatorname{Ber} g_{AB}}|D(z)|\,, \gamma_A = -\partial_A \log\rho(z) = -(-1)^B\Gamma^B_{BA}\,. \qquad (37)$$

Here g_{AB} is a covariant tensor defining Riemannian structure, ($S^{AB} = g^{AB}$) and Γ^A_{BC} are Christoffel symbols of Levi-Civita connection of this Riemannian structure. $\operatorname{Ber} g_{AB}$ is Berezinian (superdeteriminant) of the matrix g_{AB}. It is super analog of determinant. The matrix g_{AB} is $n|m\times n|m$ even matrix and its Berezinian is given by the formula

$$\operatorname{Ber} g_{AB} = \operatorname{Ber}\begin{pmatrix} g_{ab} & g_{a\beta}\\ g_{\alpha b} & g_{\alpha\beta}\end{pmatrix} = \det\left(\frac{g_{ab}-g_{a\gamma}g^{\gamma\delta}g_{\delta b}}{\det g_{\alpha\beta}}\right)\,. \qquad (38)$$

(Here as usual $g^{\gamma\delta}$ stands for the matrix inverse to the matrix $g_{\gamma\delta}$.)

The situation is essentially different in the case if $\mathbf{S} = S^{AB}$ is an odd supersymmetric contravariant tensor field and respectively $\Delta = S^{AB}\partial_A\partial_B + \dots$ is an odd operator. In this case one comes naturally to the odd Poisson structure on supermanifold M if tensor $\mathbf{S}$ obeys additional conditions.

Namely, consider cotangent bundle T^*M to supermanifold M with local coordinates (z^A, p_B) where p_A are coordinates in fibres dual to coordinates z^A ($p_A \sim \frac{\partial}{\partial z^A}$). Supersymmetric contravariant tensor field $\mathbf{S} = S^{AB}$ defines quadratic master-Hamiltonian, odd function $H_{\mathbf{S}} = \frac{1}{2}S^{AB}p_Ap_B$ on cotangent bundle T^*M. This quadratic master-Hamiltonian defines the odd bracket on the functions on M as a derived bracket:

$$\{f,g\} = ((f, H_{\mathbf{S}})\,, g)), \qquad p\left(\{f,g\}\right) = p(f) + p(g) + 1\,. \qquad (39)$$

Here (,) is canonical Poisson bracket on the cotangent bundle T^*M. The odd derived bracket is anti-commutative with respect to shifted parity and it obeys Leibnitz rule:

$$\{f,g\} = -(-1)^{(p(f)+1)(p(g)+1)}\{g,f\}, \quad \{f,gh\} = \{f,g\}h + (-1)^{p(g)p(h)}\{f,h\}g\,.$$

This odd derived bracket becomes *an odd Poisson bracket* in the case if it obeys Jacobi identity

$$(-1)^{p((f)+1)p((h)+1)}\{\{f,g\},h\} + (-1)^{p((g)+1)p((f)+1)}\{\{g,h\},f\} + (-1)^{p((h)+1)p((g)+1)}\{\{h,f\},g\} = 0\,. \qquad (40)$$

It is a beautiful fact that *the condition that derived bracket* (39) *obeys Jacobi identity can be formulated as a quadratic condition* $(H,H)=0$ *for the master-Hamiltonian*:

$$(H_{\mathbf{S}},H_{\mathbf{S}})=0 \Leftrightarrow \text{Jacobi identity for the derived bracket } \{\,,\,\} \text{ holds.} \tag{41}$$

(See for detail [14]). In the case if $\mathbf{S}$ is an even field (Riemannian geometry) master-Hamiltonian H is even function and Jacobi identity is trivial (see for detail [14] and [15].)

From now on suppose that tensor field $\mathbf{S}$ is odd and it defines an odd Poisson bracket on the supermanifold M, i.e. the relation (41) holds. This odd Poisson bracket corresponds to an odd symplectic structure in the case if the bracket is non-degenerate, i.e. the odd tensor field $\mathbf{S}$ is non-degenerate tensor field. The condition of non-degeneracy means that there exists inverse covariant tensor field S_{BC}: $S^{AB}S_{BC}=\delta^B_C$. Since the matrix S^{AB} is an odd matrix ($p(\mathbf{S}^{AB})=p(A)+p(B)+1$) this implies that matrix S^{AB} has equal number of even and odd dimensions. We come to conclusion that for an odd symplectic supermanifold even and odd dimensions have to coincide. It is necessarily $n|n$-dimensional.

The basic example of an odd symplectic supermanifold is the following: for an arbitrary usual manifold M consider its cotangent bundle T^*M and change parity of the fibres in this bundle. We come to an odd symplectic supermanifold ΠT^*M. To arbitrary local coordinates x^a on M one can associate local coordinates (x^a,θ_a) in ΠT^*M, where odd coordinates θ_a transform as ∂_a:

$$x^{a'}=x^{a'}(x^a)\,,\qquad \theta_{a'}=\frac{\partial x^a}{\partial x^{a'}}\theta_a\,. \tag{42}$$

In these local coordinates the non-degenerate odd Poisson bracket is well-defined by the relations

$$\{x^a,\theta_b\}=\delta^a_b,\ \{x^a,x^b\}=0,\ \{\theta_a,\theta_b\}=0\,. \tag{43}$$

(These relations are invariant with respect to coordinate transformations (42).)

Remark 9. An arbitrary odd symplectic supermanifold E is symplectomorphic to cotangent bundle of a usual manifold M. One may take M as even Lagrangian surface in M. (See for detail [12].) One can consider instead supermanifold E the cotangent bundle ΠT^*M for usual manifold M. The *difference* between cotangent bundle to M with changed parity of fibres and supermanifold ΠT^*M is that in the supermanifold ΠT^*M one may consider arbitrary parity preserving coordinate transformations of local coordinates x and θ which may destroy vector bundle structure, not only the transformations (42) which preserve the structure of vector bundle.

4.5. **Groupoid $C_{\mathbf{S}}$ for an odd symplectic supermanifold.** Let E be $(n|n)$-dimensional odd symplectic supermanifold, where an odd symplectic structure and respectively odd non-degenerate Poisson structure are defined by contravariant supersymmetric non-degenerate odd tensor field $\mathbf{S}=S^{AB}$ such that Jacobi identities (40) hold. We study second order odd operators $\Delta=\frac{1}{2}S^{AB}+\dots$ of weight $\delta=0$[4].

Let $\boldsymbol{\gamma}$ be an arbitrary connection on volume forms. The differential operator $\Delta=\Delta_{\text{sing}}(\mathbf{S},\boldsymbol{\gamma})$ of weight $\delta=0$ with principal symbol $\mathbf{S}$ defined by equation (32) transforms

[4]The following construction of groupoid is obviously valid in the general Poisson case, but in this subsection we are mainly interested in the odd symplectic case

half-densities to half-densities. Due to the formulae (32), (33) and (36) this operator equals to

$$\Delta_{\text{sing}}(\mathbf{S},\boldsymbol{\gamma})\colon \mathcal{F}_{\frac{1}{2}}\to\mathcal{F}_{\frac{1}{2}},\qquad \Delta_{\text{sing}}(\mathbf{S},\boldsymbol{\gamma})=\frac{1}{2}\left(S^{AB}\partial_B\partial_A+\partial_B g^{BA}\partial_a+\frac{1}{2}\partial_A\gamma^A-\frac{1}{4}\gamma_A\gamma^A\right). \tag{44}$$

We come to the groupoid

$$C_{\mathbf{S}}=\left\{\text{Groupoid of arrows }\boldsymbol{\gamma}\xrightarrow{\mathbf{X}}\boldsymbol{\gamma}'\text{ such that }\operatorname{div}_{\boldsymbol{\gamma}}\mathbf{X}-\frac{1}{2}\mathbf{X}^2=0\right\}$$

and to the operator on half-densities depending on the class of connections

$$\Delta([\boldsymbol{\gamma}])=\frac{1}{2}\left(S^{AB}\partial_B\partial_A+\partial_B g^{BA}\partial_a+U_{\mathbf{S}}([\boldsymbol{\gamma}])\right),\ \text{where}\ U_{\mathbf{S}}([\boldsymbol{\gamma}])=\frac{1}{2}\partial_A\gamma^A-\frac{1}{4}\gamma_A\gamma^A. \tag{45}$$

It is here where a similarity with Riemannian case finishes. On Riemannian manifold one can consider canonical volume form and distinguished Levi-Civita connection which induces canonical flat connection $\boldsymbol{\gamma}$ (see equation (37)). On an odd symplectic supermanifold there is no canonical volume form [5] and there is no distinguished connection on vector fields. On the other hand it turns out that in this case one can construct the class of distinguished connections which belong to an orbit of groupoid $C_{\mathbf{S}}$. Namely study the equation

$$\operatorname{div}_{\boldsymbol{\gamma}}\mathbf{X}-\frac{1}{2}\mathbf{X}^2=0, \tag{46}$$

which defines the groupoid $C_{\mathbf{S}}$. According to equations (34), (44) and (45) we see that for operators $\Delta([\boldsymbol{\gamma}])$ acting on half-densities we have that

$$\Delta([\boldsymbol{\gamma}'])-\Delta([\boldsymbol{\gamma}])=\Delta_{\text{sing}}(\mathbf{S},\boldsymbol{\gamma}')-\Delta_{\text{sing}}(\mathbf{S},\boldsymbol{\gamma})=\frac{1}{4}\left(\operatorname{div}_{\boldsymbol{\gamma}}\mathbf{X}-\frac{1}{2}\mathbf{X}^2\right). \tag{47}$$

We call the equation (46) *Batalin-Vilkovisky equation.* Study this equation.

It is convenient to work in *Darboux coordinates.* Local coordinates $z^A=(x^a,\theta_b)$ on supermanifold E are called Darboux coordinates if non-degenerate odd Poisson bracket has the appearance (43) in these coordinates.

We say that connection $\boldsymbol{\gamma}$ is Darboux flat if it vanishes in some Darboux coordinates.

Lemma 1. *Let $\boldsymbol{\gamma},\boldsymbol{\gamma}'$ be two connections such that both connection $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}'$ are Darboux flat. Then the arrow $\boldsymbol{\gamma}\xrightarrow{\mathbf{X}}\boldsymbol{\gamma}'$ belongs to the groupoid $C_{\mathbf{S}}$. i.e. the Batalin-Vilkovisky equation $\operatorname{div}_{\boldsymbol{\gamma}}\mathbf{X}-\frac{1}{2}\mathbf{X}^2=0$ holds for covector field $\mathbf{X}=\boldsymbol{\gamma}'-\boldsymbol{\gamma}$.*

We will prove this lemma later.

Remark 10. In fact lemma implies that a class of locally defined Darboux flat connections defines globally the pseudoscalar function $U_{\mathbf{S}}$ in (44). Let $\{z^A_{(\alpha)}\}$ be an arbitrary atlas of Darboux coordinates on E. We say that the collection of local connections $\{\boldsymbol{\gamma}_{(\alpha)}\}$ is adjusted to Darboux atlas $\{z^A_{(\alpha)}\}$ if every local connection $\boldsymbol{\gamma}_{(a)}$ (defined in the chart $z^A_{(\alpha)}$) vanishes in these local Darboux coordinates $z^A_{(\alpha)}$. Let $\{\boldsymbol{\gamma}_{(\alpha)}\}$ and $\{\boldsymbol{\gamma}'_{(\alpha')}\}$ be two families of local connections adjusted to Darboux atlases $\{z^A_{(\alpha)}\}$ and $\{z^{A'}_{(\alpha')}\}$ respectively. Then due to Lemma all arrows $\boldsymbol{\gamma}_{(\alpha)}\xrightarrow{\mathbf{X}}\boldsymbol{\gamma}_{(\alpha')}$ $\boldsymbol{\gamma}'_{(\alpha)}\xrightarrow{\mathbf{X}}\boldsymbol{\gamma}'_{(\alpha')}$ and $\boldsymbol{\gamma}_{(\alpha)}\xrightarrow{\mathbf{X}}\boldsymbol{\gamma}'_{(\alpha')}$ belong to local groupoid $C_{\mathbf{S}}$ (if charts $(z^A_{(\alpha)})$, $(z^A_{(\alpha')})$, $(z^{A'}_{(\alpha)})$ and $(z^{A'}_{(\alpha')})$ intersect). This means that in spite of the fact that the family $\{\gamma_\alpha\}$ does

[5] Naive generalisation of formulae (37) and (38) does not work since in particular S^{AB} is not an even matrix

not define the global connection, still equations (46) hold locally and operator $\Delta = \Delta(\mathbf{S}, \boldsymbol{\gamma}_\alpha)$ globally exists. (These considerations for locally defined groupoid can be performed for arbitrary case. One can consider the family of locally defined connections $\{\boldsymbol{\gamma}_a\}$ such that they define global operator (32).) On the other hand in a case of an odd symplectic supermanifold there exists a global Darboux flat connection, i.e. the connection $\boldsymbol{\gamma}$ such in a vicinity of an arbitrary point this connection vanishes in some Darboux coordinates. Show it.

Without loss of generality suppose that $E = \Pi T^*M$ (see Remark (9).) Let σ be an arbitrary volume form on M (we suppose that M is orientable). Choose an atlas $\{x^a_{(\alpha)}\}$ of local coordinates on M such that σ is the coordinate volume form, i.e. $\sigma = dx^1_{(\alpha)} \wedge \dots dx^n_{(\alpha)}$. Then consider associated atlas $\{x^a_{(\alpha)}, \theta_a(\alpha)\}$ in supermanifold ΠT^*M which is an atlas of Darboux coordinates. For this atlas as well as for the atlas $\{x^a_{(\alpha)}\}$ Jacobians of coordinate transformations are equal to 1. Thus we constructed atlas of *special Darboux coordinates* in which all the Jacobians of coordinate transformations are equal to 1. The coordinate volume form $\rho = D(x,\theta)$ is globally defined. The flat connection defined by this volume form vanishes. We defined globally Darboux flat connection.

We come to Proposition

Proposition 5. *In an odd symplectic supermanifold there exists a canonical orbit of connections. It is the class $[\boldsymbol{\gamma}]$ in the groupoid $C_{\mathbf{S}}$, where $\boldsymbol{\gamma}$ is* an arbitrary Darboux flat connection. *We will call this canonical class of connections "the class of Darboux flat connections".*

For any connection belonging to the canonical orbit of connections, the pseudoscalar function $U_{\mathbf{S}}$ in (45) vanishes in arbitrary Darboux coordinates[6]. The operator $\Delta = \Delta[\boldsymbol{\gamma}]$ on half-densities corresponding to this class of connections has the following appearance in arbitrary Darboux coordinates $z^A = (x^a, \theta_b)$:

$$\Delta = \frac{\partial^2}{\partial x^a \partial \theta_a} . \tag{48}$$

(This canonical operator on half-densities was introduced in [12].)

Now prove Lemma 1.

For an arbitrary volume form $\boldsymbol{\rho}$ consider an operator

$$\Delta_{\boldsymbol{\rho}} f = \frac{1}{2} \mathrm{div}\,_{\boldsymbol{\rho}} \mathrm{grad}\, f . \tag{49}$$

Here grad f is Hamiltonian vector field $\{f, z^A\}\frac{\partial}{\partial z^A}$ corresponding to the function f. (Compare with (2).) This is the famous Batalin-Vilkovisky odd Laplacian on functions. In the case if $z^A = (x^a, \theta_a)$ are Darboux coordinates and a volume form $\boldsymbol{\rho}$ is the coordinate volume form, i.e. $\boldsymbol{\rho} = D(x,\theta)$, then odd Laplacian in these Darboux coordinates has the appearance

$$\Delta = \frac{\partial^2}{\partial x^a \partial \theta_a} . \tag{50}$$

(This is the initial form of the Batalin-Vilkovisky operator in [3]. (Geometrical meaning of BV operator, and how formulae (49) and (50) are related with canonical operator (48) on semidensities see in [10, 19, 12].)

The equation (46) characterising the groupoid (the Batalin-Vilkovisky equation) is related with the Batalin-Vilkovisky operator by the following identity:

$$-e^{\frac{F}{2}} \Delta_{\boldsymbol{\rho}} e^{\frac{-F}{2}} = \frac{1}{4} \mathrm{div}\,_{\boldsymbol{\gamma}} \mathbf{X} - \frac{1}{8} \mathbf{X}^2 , \tag{51}$$

[6]This function vanishes not only for globally defined Darboux flat connection but for a family of connections adjusted to an arbitrary Darboux atlas (see Remark 10)

where connection $\boldsymbol{\gamma}$ is a flat connection induced by volume form ($\gamma_a = -\partial_a \log \rho$) and vector field $\mathbf{X}$ is Hamiltonian vector field of the function F.

We use this identity to prove the lemma. Let connection $\boldsymbol{\gamma}$ vanishes in Darboux coordinates $z^A = (x^a, \theta_a)$ and connection $\boldsymbol{\gamma}'$ vanishes in Darboux coordinates $z^{A'} = (x^{a'}, \theta_{a'})$. Then (compare with equation (11)) connection $\boldsymbol{\gamma}'$ has in Darboux coordinates $z^A = (x^a, \theta_a)$ the following appearance

$$\gamma'_A = \frac{\partial z^{A'}(z^A)}{\partial z^A} \left(\gamma_{A'} + \partial_{A'} \log J\right) \,,$$

where J is Jacobian of Darboux coordinates transformation $J = \mathrm{Ber}\ \ J = \mathrm{Ber}\, \frac{\partial(x,\theta)}{\partial(x',\theta')}$ (see also the formula (38))

Hence for the arrow $\boldsymbol{\gamma} \xrightarrow{\mathbf{X}} \boldsymbol{\gamma}'$ the covector field $\mathbf{X}$ equals to $X_A = -\frac{\partial}{\partial z^a} \log \mathrm{Ber}\, \frac{\partial(x',\theta')}{\partial(x,\theta)}$.

Apply identity (51) where $\gamma_a = 0$, $\boldsymbol{\rho} = D(x,\theta)$ is coordinate volume form and $F = -\log \mathrm{Ber}\, \frac{\partial(x',\theta')}{\partial(x,\theta)}$. Using (49) and (50) we arrive at

$$\frac{1}{4}\mathrm{div}_{\,\boldsymbol{\gamma}}\mathbf{X} - \frac{1}{8}\mathbf{X}^2 = -e^{\frac{F}{2}} \Delta_{\boldsymbol{\rho}} e^{\frac{-F}{2}} = -\left(\sqrt{\mathrm{Ber}\, \frac{\partial(x,\theta)}{\partial(x',\theta')}}\right) \frac{\partial^2}{\partial x^a \partial \theta_a} \left(\sqrt{\mathrm{Ber}\, \frac{\partial(x',\theta')}{\partial(x,\theta)}}\right) = 0\,.$$

The last identity is the famous Batalin-Vilkovisky identity [4] which stands in the core of the geometry of Batalin-Vilkovisky operator.

Lemma is proved.

Remark 11. The canonical operator (48) assigns to every even non-zero half density $\mathbf{s}$ and to every volume form $\boldsymbol{\rho}$ the functions $\sigma_{\mathbf{s}}$ and $\sigma_{\boldsymbol{\rho}}$:

$$\sigma(\mathbf{s}) = \frac{\Delta \mathbf{s}}{\mathbf{s}}, \qquad \sigma(\boldsymbol{\rho}) = \frac{\Delta \sqrt{\boldsymbol{\rho}}}{\boldsymbol{\rho}}$$

(see [12]). In the articles [1, 2] Batalin and Bering considered geometrical properties of the canonical operator (48) on semidensities. In these considerations they used the formula for expressing the canonical operator (48) in arbitrary coordinates. This formula was suggested by Bering in [6]. Clarifying geometrical meaning of this formula and analysing the geometrical meaning of the scalar function $\sigma(\boldsymbol{\rho})$ they come to beautiful result: if ∇ is an arbitrary torsion-free affine connection in an odd symplectic supermanifold which is compatible with volume form $\boldsymbol{\rho}$, then the scalar curvature of this connection equals (up to a coefficient) to the function $\sigma_{\boldsymbol{\rho}}$.

Remark 12. In work [14] we considered in particular the following "Batalin-Vilkovisky groupoid" of volume forms on an odd Poisson manifold: the arrows $\boldsymbol{\rho} \xrightarrow{J} \boldsymbol{\rho}'$, where $J = \frac{\boldsymbol{\rho}'}{\boldsymbol{\rho}}$, are defined by the Batalin-Vilkovisky equation $\Delta_{\boldsymbol{\rho}} \sqrt{J} = 0$. The operator $\Delta_{\boldsymbol{\rho}}$ is defined by equation (49). Assign to each arrow $\boldsymbol{\rho} \xrightarrow{J} \boldsymbol{\rho}'$ the arrow $\boldsymbol{\gamma} \xrightarrow{\mathbf{X}} \boldsymbol{\gamma}'$ of the groupoid $C_{\mathbf{S}}$ such that the connections $\boldsymbol{\gamma}, \boldsymbol{\gamma}'$ are defined by volume forms $\boldsymbol{\rho}, \boldsymbol{\rho}'$ respectively ($\gamma_a = -\partial_a \log \rho$ and $\gamma'_a = -\partial_a \log \rho'$). Then it follows from equation (51) that the groupoid of volume forms is a subgroupoid of the groupoid $C_{\mathbf{S}}$.

Both the Batalin–Vilkovisky groupoid and the groupoid of connections $C_{\mathbf{S}}$ considered here can be regarded as Lie groupoids over infinite-dimensional manifolds, which are the space $\mathrm{Vol}^{\times}(M)$ of the non-degenerate volume forms and the space $\mathrm{Conx}(M)$ of the connections on densities on a manifold M respectively. The corresponding Lie algebroids can be described as follows.

For the Batalin–Vilkovisky groupoid, the Lie algebroid is the vector bundle over the (infinite-dimensional) manifold $\mathrm{Vol}^{\times}(M)$ whose the fiber over the point $\boldsymbol{\rho}$ is the vector space of all solutions of the equation $\Delta_{\boldsymbol{\rho}} F = 0$ where $F \in C^{\infty}(M)$. The anchor is tautological: it sends a function F to the infinitesimal shift $\boldsymbol{\rho} \mapsto \boldsymbol{\rho} + \varepsilon \boldsymbol{\rho} F$. A section of this bundle is a functional $F[\boldsymbol{\rho}]$ of a volume form with values in functions on M such that for each $\boldsymbol{\rho}$, the above equation is satisfied. The Lie bracket is the restriction of the canonical commutator of vector fields on $\mathrm{Vol}^{\times}(M)$ and can be expressed by an explicit formula

$$[F, G][\boldsymbol{\rho}; x] = \int_M Dy\, \rho(y) \left(F[\boldsymbol{\rho}; y] \frac{\delta G[\boldsymbol{\rho}; x]}{\delta \rho(y)} - G[\boldsymbol{\rho}; y] \frac{\delta F[\boldsymbol{\rho}; x]}{\delta \rho(y)} \right) .$$

Here we write $F[\boldsymbol{\rho}; x]$ for the value of $F[\boldsymbol{\rho}] \in C^{\infty}(M)$ at $x \in M$.

For the groupoid of connections (with a fixed tensor density S^{ab} of weight δ), the Lie algebroid is the vector bundle over $\mathrm{Conx}(M)$ whose fiber over $\boldsymbol{\gamma} \in \mathrm{Conx}(M)$ is the vector space of all solutions of the equation $\mathrm{div}_{\boldsymbol{\gamma}} \mathbf{X} = 0$. A section is a functional of a connection taking values in these vector spaces. The anchor is tautological: it sends a covector field X_a to the infinitesimal shift of the connection $\gamma_a \mapsto \gamma_a + \varepsilon X_a$. The Lie bracket can be expressed by the formula

$$[X, Y]_a[\boldsymbol{\gamma}; x] = \int_M Dy \left(X_b[\boldsymbol{\gamma}; y] \frac{\delta Y_a[\boldsymbol{\gamma}; x]}{\delta \gamma_b(y)} - Y_b[\boldsymbol{\gamma}; y] \frac{\delta X_a[\boldsymbol{\gamma}; x]}{\delta \gamma_b(y)} \right) .$$

(For supermanifolds the formulas for the brackets contain extra signs.) Note that since the groupoids in question are subgroupoids of the trivial (pair) groupoids, these Lie algebroids are subalgebroids of the respective tangent bundles.

4.6. **Groupoid $C_{\mathbf{S}}$ for the line.** We return here to simplest possible manifold—real line. The symmetric tensor field $\mathbf{S}$ of rank 2 and of weight δ on real line $\mathbb{R}$ is a density of the weight $\delta - 2$: $\mathbf{S} = S\partial_x^2 |D(x)|^{\delta} \sim S|D(x)|^{\delta-2}$. Consider on $\mathbb{R}$ the canonical vector density $|D(x)|\partial_x$ which is invariant with respect to change of coordinates. Its square defines canonical tensor density $\mathbf{S}_{\mathbb{R}} = |D(x)|^2 (\partial_x)^2$ of weight $\delta = 2$.

We see that on the line there is a canonical pencil of second order operators of the weight $\delta = 2$: $|D(x)|^2 \left(\partial_x^2 + \dots\right)$ with canonical principal symbol $\mathbf{S}_{\mathbb{R}} = |D(x)|^2 (\partial_x)^2$. The operator (32) belonging to this pencil acts on densities of weight $\frac{1-\delta}{2} = -\frac{1}{2}$ and transforms them into densities of weight $\frac{1+\delta}{2} = \frac{3}{2}$. According to (32) It has the following appearance:

$$\Delta(\boldsymbol{\gamma})\colon \quad \Psi(x)|Dx|^{-\frac{1}{2}} \mapsto \Phi(x)|D(x)|^{\frac{3}{2}} = \frac{1}{2} \left(\frac{\partial^2 \Psi(x)}{\partial x^2} + U(x)\Psi(x) \right) |Dx|^{\frac{3}{2}} ,$$

where according to the equation (33)

$$U_{\boldsymbol{\gamma}}(x) = -\frac{1}{4} \left(\gamma_x + \frac{1}{2}\gamma^2 \right) |D(x)|^2. \tag{52}$$

This is Sturm-Lioville operator recognisable by speciaialists in projective geometry and integrable systems[7] (see e.g. [9] or the book [18]).

[7]The operator Δ corresponds to a curve $t \mapsto [u_1(t) : u_2(t)]$, $\mathbb{R} \to \mathbb{R}P^1$ in projective line defined by the solutions of equation $\Delta u = 0$.

We see that in this case the difference of operators is

$$\Delta(\boldsymbol{\gamma}') - \Delta(\boldsymbol{\gamma}) = -\frac{1}{4}\left(\gamma'_x + \frac{1}{4}(\gamma')^2\right)|D(x)|^2 + \frac{1}{4}\left(\gamma_x + \frac{1}{2}(\gamma)^2\right)|D(x)|^2 =$$

$$-\frac{1}{4}\left(\operatorname{div}\mathbf{X} + \frac{1}{2}\mathbf{X}^2\right).$$

Here $\mathbf{X} = (\gamma' - \gamma)|D(x)|^2\partial_x$ is vector density of the weight $\delta = 2$. (compare with formulae(34) and (47)).

Using formulae (35) we come to the following canonical groupoid $C_{\mathbb{R}}$ on the line:

$$C_{\mathbb{R}} = \{\text{Groupoid of arrows } \boldsymbol{\gamma} \xrightarrow{\mathbf{X}} \boldsymbol{\gamma}' \text{such that } \Delta(\boldsymbol{\gamma}') = \Delta(\boldsymbol{\gamma}), i.e.\, U_{\boldsymbol{\gamma}'} = U_{\boldsymbol{\gamma}}\} =$$

$$= \{\text{Groupoid of arrows } \boldsymbol{\gamma} \xrightarrow{\mathbf{X}} \boldsymbol{\gamma}' \text{ such that } \operatorname{div}_{\boldsymbol{\gamma}}\mathbf{X} + \frac{1}{2}\mathbf{X}^2 = 0\},$$

where $\Delta(\boldsymbol{\gamma})$ is the Sturm-Lioville operator (52). It depends on the orbit of connection $\boldsymbol{\gamma}$, the class $[\boldsymbol{\gamma}]$.

Analyse the equation $\operatorname{div}\mathbf{X} + \frac{1}{2}\mathbf{X}^2 = 0$ defining the canonical groupoid $C_{\mathbb{R}}$ and compare it with the cocycle related with the operator.

If covector field equals to $\boldsymbol{\gamma}' - \boldsymbol{\gamma} = a(x)dx$, then vector density equals to $\mathbf{S}_{\mathbb{R}}(a(x)dx) = a(x)|D(x)|^2\partial_x$. Hence $\mathbf{X}^2 = a^2(x)|D(x)|^2$ and $\operatorname{div}\mathbf{X} = (a_x + \gamma a)|D(x)|^2$. We come to the equation:

$$\operatorname{div}\mathbf{X} + \frac{1}{2}\mathbf{X}^2 = \left(a_x + \gamma a + \frac{1}{2}a^2\right)|D(x)|^2 = 0\,.$$

Solve this differential equation. Choose coordinate such that γ vanishes in this coordinate. Then

$$\mathbf{X} = \frac{2dx}{C + x}, \qquad \text{where } C \text{ is a constant}\,. \tag{53}$$

On the other hand analyze the action of diffeomorphisms on the connection $\boldsymbol{\gamma}$ and the Sturm-Lioville operator (52). Let $f = f(x)$ be a diffeomorphism of $\mathbb{R}$. (We consider compactified $\mathbb{R} \sim S^1$ and diffeomorphisms preserving orientation.) The new connection $\boldsymbol{\gamma}^{(f)}$ equals to $y_x\left(\boldsymbol{\gamma}\big|_{y(x)} + (\log x_y)_x\right)dx$ and the covector field $\boldsymbol{\gamma}^{(f)} - \boldsymbol{\gamma}$ equals to

$$\mathbf{X}^{(f)} = \boldsymbol{\gamma}^{(f)} - \boldsymbol{\gamma} = \gamma(y(x))dy + (\log x_y)_y\, dy - \gamma(x)dx\,.$$

We come to cocycle on group of diffeomorphisms:

$$c_{\boldsymbol{\gamma}}(f) = \Delta^f(\boldsymbol{\gamma}) - \Delta(\boldsymbol{\gamma}) = \Delta(\boldsymbol{\gamma}^f) - \Delta(\boldsymbol{\gamma}) = \frac{1}{4}\left(U_{\boldsymbol{\gamma}^f} - U_{\boldsymbol{\gamma}}\right) =$$

$$-\frac{1}{4}\left(\operatorname{div}\mathbf{X}^{(f)} + \frac{1}{2}\left(\mathbf{X}^{(f)}\right)^2\right). \tag{54}$$

In coordinate such that $\gamma = 0$, $\mathbf{X}^{(f)} = (\log x_y)_y\, dy$. Combining with a solution (53) we come to equation $(\log x_y)_x = \frac{2}{C+x}$. Solving this equation we see that

$$\operatorname{div}\mathbf{X}^{(f)} + \frac{1}{2}\left(\mathbf{X}^{(f)}\right)^2 = 0 \Leftrightarrow y = \frac{ax+b}{cx+d} \text{ is a projective transformation}\,.$$

The cocycle (54) is coboundary in the space of second order operators and it is a non-trivial cocycle in the space of densities of the weight 2. This cocycle vanishes on projective transformations. This is well-known cocycle related with Schwarzian derivative (see the book [18] and citations there):

$$c_{\gamma}(f) = \Delta^f(\boldsymbol{\gamma}) - \Delta(\boldsymbol{\gamma}) = \Delta(\boldsymbol{\gamma}^f) - \Delta(\boldsymbol{\gamma}) = \frac{1}{2}\left(U_{\boldsymbol{\gamma}^f} - U_{\boldsymbol{\gamma}}\right) =$$

$$-\frac{1}{4}\left(\operatorname{div}\mathbf{X}^{(f)} + \frac{1}{2}\left(\mathbf{X}^{(f)}\right)^2\right) = -\frac{1}{4}\left(U_{\boldsymbol{\gamma}}(y)|D(y)|^2 + \mathcal{S}\left(x(y)\right)|D(y)|^2 - U_{\boldsymbol{\gamma}}(x)|D(x)|^2\right),$$

where

$$\mathcal{S}(x(y)) = \frac{x_{yyy}}{x_y} - \frac{3}{2}\left(\frac{x_{yy}}{x_y}\right)^2 .$$

is *Schwarzian* of the transformation $x = x(y)$. If $\gamma = 0$ in coordinate x then $c(f) = \mathcal{S}(x(y))|D(y)|^2$.

4.7. **Invariant densities on $1|1$-codimension submanifolds in an odd symplectic supermanifold and mean curvature.** In the previous examples we considered second order operators which depend on a class of connections on volume forms. In particular we considered for odd symplectic supermanifold the canonical class of Darboux flat connections (see the Proposition 5) and with use of this class redefined the canonical operator (48).

Now we consider an example of geometrical constructions which depend on second order derivatives and on a class *of Darboux flat affine connections.*

Let E be an odd symplectic supermanifold equipped with volume form $\boldsymbol{\rho}$. Let C be a non-degenerate submanifold of codimension $(1|1)$ in E (induced Poisson structure on C is non-degenerate). We call such a submanifold "hypersurface".

For an arbitrary affine connection ∇ and arbitrary vector field Ψ consider the following object:

$$A(\nabla, \Psi) = \operatorname{Tr}\left(\Pi\left(\nabla\Psi\right)\right) - \operatorname{div}_{\rho}\Psi\,, \tag{55}$$

where Π is the projector on $(1|1)$-dimensional planes which are symplectoorthogonal to hypersurface C at points of this hypersurface. (We define these objects in a vicinity of C.)

Let vector field Ψ be symplectoorthogonal to the hypersurface C at points of C. Then one can see that at points of C

$$A(\nabla, f\Psi) = fA(\nabla, \Psi) \tag{56}$$

for an arbitrary function f. Thus $A(\nabla, \Psi)$ is well-defined on C in the case if Ψ is a vector field defined only at C and Ψ is symplectoorthogonal to C. This object is interesting since it is related with canonical vector valued half-density and canonical scalar half-density on the manifold C (see for detail [11].)

Namely let Ψ be a vector field on hypersurface C symplectoorthogonal to C. From now on we suppose that it also obeys to following additional conditions

- it is an odd vector field $p(\Psi = \Psi^A\partial_A) = p(\Psi^A) + p(A) = 1$,
- it is non-degenerate, i.e. at least one of components is not-nilpotent,
- $\omega(\Psi, \Psi) = 0$, where ω is the symplectic form in E, defining its symplectic structure.

One can see that these conditions uniquely define vector field Ψ at every point of C up to a multiplier function.

Consider now a following volume form ρ_Ψ on C: Let $\mathbf{H}$ be an even vector field on hypersurface C such that it is symplectoorthogonal to C and $\omega(\mathbf{H},\Psi)=1$. Define an half-density $\boldsymbol{\rho}_\Psi$ by the condition that for an arbitrary basis $\{\mathbf{e}_1,\dots \mathbf{e}_{n-1};\mathbf{f}_1,\dots,\mathbf{f}_{n-1}\}$ of surface C

$$\boldsymbol{\rho}_\Psi\left(\mathbf{e}_1,\dots \mathbf{e}_{n-1};\mathbf{f}_1,\dots,\mathbf{f}_{n-1}\right)=\boldsymbol{\rho}\left(\mathbf{e}_1,\dots \mathbf{e}_{n-1},\mathbf{H};\mathbf{f}_1,\dots,\mathbf{f}_{n-1},\Psi\right)\,.$$

(Here $\mathbf{e}_1,\dots \mathbf{e}_{n-1}$ are even basis vectors and $\mathbf{f}_1,\dots,\mathbf{f}_{n-1}$ are odd basis vectors.) Using formula (38) for Berezinian and relation (56) one can see that for an arbitrary function f,

$$\boldsymbol{\rho}_{f\psi}=\frac{\boldsymbol{\rho}_\Psi}{f^2}\,.$$

We come to conclusion that vector valued half-density $\Psi\sqrt{\boldsymbol{\rho}_\Psi}$ is well-defined odd half-density on hypersurface C. Applying equation (55) we come to well-defined half-density on hypersurface C: $\mathbf{s}_{C}(\nabla)=A(\nabla,\Psi)\sqrt{\boldsymbol{\rho}_\Psi}$. This half-density depends only on affine connection ∇.

We say that affine supersymmetric connection ∇ on E with Christoffel symbols Γ^C_{AB} is Darboux flat affine if there exist Darboux coordinates $z^A=(x^a,\theta_b)$ such that in these Darboux coordinates the Chrsitoffel symbols of the connection vanish: $\nabla_A\partial_B=0$. (Darboux flat affine connection on E induces Darboux flat connection $\boldsymbol{\gamma}$: $\gamma_A=(-1)^B\Gamma^B_{AB}$ on volume forms.)

Proposition 6. *The half-density $\mathbf{s}_{C}(\nabla)$ does not depend on a connection in the class of Darboux flat connections: $\mathbf{s}_{C}(\nabla)=\mathbf{s}_{C}(\nabla')$ for two arbitrary Darboux flat affine onnections ∇ and ∇'.*

This statement in not explicit way in fact was used in the work [11] where the half-density was constructed in Darboux coordinates.

The Proposition implies the existence of canonical half-density on hyperLsurfaces in odd symplectic supermanifold. This semidensity was first calculated straightforwardly in [13]

On one hand the invariant semindensity in odd symplectic supermanifold is an analogue of Poincare-Cartan integral invariants. On the other hand the constructions above are related with mean curvature of hypersurfaces (surfaces of codimension 1) in the even Riemannian case: if C is surface of codimension $(1|0)$ in Riemannian manifold M then one can consider the canonical Levi-Civita connection and canonical volume form. Applying constructions above we come to mean curvature. In the odd symplectic case there is no preferred affine connection compatible with the symplectic structure(see for detail [11]).

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