ON GENERALIZED SYMMETRIC POWERS AND A GENERALIZATION OF KOLMOGOROV–GELFAND–BUCHSTABER–REES THEORY

H. M. KHUDAVERDIAN AND TH. TH. VORONOV

In 1939, I. M. Gelfand and A. N. Kolmogorov showed [1] that for a compact Hausdorff topological space X, homomorphisms of the algebra of continuous functions $C(X)$ to the field of real numbers are in a one-to-one correspondence with points of X. The algebra $C(X)$ is considered without a topology; this result is less known than its analog that gave birth to the theory of normed rings. The Gelfand–Kolmogorov theorem may be viewed as a description of the image of the canonical embedding of X into the infinite-dimensional linear space $V = A^*$, where $A = C(X)$, by a system of quadratic equations: $\mathbf{f}(1) = 1$, $\mathbf{f}(a)^2 - \mathbf{f}(a^2) = 0$, indexed by elements of A. This aspect was recently emphasized by V. M. Buchstaber and E. G. Rees (see [2] and references therein). They showed that there is a natural embedding into V not only for X, but also for all its symmetric powers $\operatorname{Sym}^n X$. To this end, algebra homomorphisms should be replaced by the so-called n-homomorphisms, and quadratic equations describing the image, by certain algebraic equations of higher degree. This theory was motivated by their earlier study of an analogue of Hopf algebra for multivalued groups. The other source is Frobenius's higher group characters.

In the present note we give a generalization of Buchstaber–Rees's theory. For a space X we construct a functorial object $\operatorname{Sym}^{p|q} X$, $p, q \geqslant 0$, and for a commutative algebra with unit A, a corresponding algebra $S^{p|q}A$. We call them 'generalized symmetric powers'. There is a canonical map from $\operatorname{Sym}^{p|q} X$ to $V = A^*$. To describe its image we introduce certain algebraic equations, extending thus the statements of Gelfand–Kolmogorov and Buchstaber–Rees. This corresponds to a description of algebra homomorphisms $S^{p|q}A \to B$ in terms of the new notion of a $p|q$-homomorphism. Our work was motivated by the results on linear operators on superspaces [3], from where comes our main tool, the 'characteristic function' of a linear map of algebras. The methods that we propose yield, in particular, a simple direct proof of the main theorem of Buchstaber and Rees.

Let A and B be commutative associative algebras with unit. Consider an arbitrary linear map $\mathbf{f}\colon A \to B$. Its *characteristic function* is defined to be $R(\mathbf{f}, a, z) = e^{\mathbf{f}\ln(1+az)}$, where $a \in A$ and z is a formal parameter. Example: if $\mathbf{f}$ is an algebra homomorphism, then $R(\mathbf{f}, a, z) = 1 + \mathbf{f}(a)z$. Algebraic properties of the map $\mathbf{f}$ are reflected in the properties of $R(\mathbf{f}, a, z)$ as a function of the variable z. The case when $R(\mathbf{f}, a, z)$ is a polynomial of degree n corresponds to the Buchstaber–Rees theory.

We call a linear map $\mathbf{f}$, a *$p|q$-homomorphism* if $R(\mathbf{f}, a, z)$ is a rational function that can be represented by the ratio of polynomials of degrees p and q. Properties of $p|q$-homomorphisms follow from general properties of $R(\mathbf{f}, a, z)$. For an arbitrary map $\mathbf{f}$, $R(\mathbf{f}, a, z)$ has the power expansion at zero $R(\mathbf{f}, a, z) = 1 + \psi_1(\mathbf{f}, a)z + \psi_2(\mathbf{f}, a)z^2 + \dots$

where $\psi_k(\mathbf{f}, a) = P_k(s_1, \ldots, s_k)$. Here $s_k = s_k(\mathbf{f}, a) = \mathbf{f}(a^k)$, and P_k are the classical Newton polynomials giving expression of elementary symmetric functions via sums of powers. The exponential property $R(\mathbf{f}+\mathbf{g}, a, z) = R(\mathbf{f}, a, z)R(\mathbf{g}, a, z)$ holds. Let $R(\mathbf{f}, a, z)$ be defined as a function of z regarded, say, as a complex variable. Consider its behaviour at infinity. By a formal transformation we can obtain $R(\mathbf{f}, a, z) = z^{\mathbf{f}(1)}e^{\mathbf{f}\ln a}e^{\mathbf{f}\ln(1+a^{-1}z^{-1})}$, cf. [3]. In particular, for $a = 1$ we have $R(\mathbf{f}, 1, z) = (1+z)^{\mathbf{f}(1)}$. Assume that $R(\mathbf{f}, a, z)$ has no essential singularity. Then $\mathbf{f}(1) = \chi \in \mathbb{Z}$ and the integer χ is the order of the pole at infinity. We arrive at the expansion near infinity $R(\mathbf{f}, a, z) = \sum_{k\leqslant\chi} \psi^*_k(\mathbf{f}, a)z^k$ where $\psi^*_k(\mathbf{f}, a) := e^{\mathbf{f}\ln a}\psi_{\chi-k}(\mathbf{f}, a^{-1})$. Denote the leading term of the expansion $e^{\mathbf{f}\ln a} =: \operatorname{ber}(\mathbf{f}, a)$ and call it, the $\mathbf{f}$-*Berezinian* of $a \in A$. Note that $a \mapsto \operatorname{ber}(\mathbf{f}, a)$ is, in general, a partially defined map $A \to B$. One can immediately see that $\mathbf{f}$-Berezinian is multiplicative: $\operatorname{ber}(\mathbf{f}, a_1a_2) = \operatorname{ber}(\mathbf{f}, a_1)\operatorname{ber}(\mathbf{f}, a_2)$. In the rational case, $\operatorname{ber}(\mathbf{f}, a)$ is the ratio of polynomials in the elements $\mathbf{f}(a^k)$.

Here are the examples to be kept in mind. If $\mathbf{f}(a) = \operatorname{tr}\boldsymbol{\rho}(a)$ for a matrix representation $\boldsymbol{\rho}\colon A \to \operatorname{Mat}(n, B)$, then $R(\mathbf{f}, a, z) = \det(1+\boldsymbol{\rho}(a)z)$ and $\operatorname{ber}(\mathbf{f}, a) = \det\boldsymbol{\rho}(a)$. For a representation by $p|q \times p|q$ matrices, we obtain $R(\mathbf{f}, a, z) = \operatorname{Ber}(1+\boldsymbol{\rho}(a)z)$. In this case, $\mathbf{f}(a) = \operatorname{str}\boldsymbol{\rho}(a)$ and $\operatorname{ber}(\mathbf{f}, a) = \operatorname{Ber}\boldsymbol{\rho}(a)$ is the ordinary Berezinian.

Multilinear symmetric functions $\Phi_k(\mathbf{f}, a_1, \ldots, a_k)$ of elements $a_i \in A$ such that $\Phi_k(\mathbf{f}, a, \ldots, a) = k!\psi_k(\mathbf{f}, a)$, satisfy the Frobenius recursion relations (see [2]). Note that for the case of a matrix representation, $s_k(\mathbf{f}, a) = \operatorname{tr}\boldsymbol{\rho}(a)^k$, $\psi_k(\mathbf{f}, a) = \operatorname{tr}\Lambda^k\boldsymbol{\rho}(a)$, and $\Phi_k(\mathbf{f}, a_1, \ldots, a_k) = k!\ \operatorname{tr}(\boldsymbol{\rho}(a_1)\wedge\ldots\wedge\boldsymbol{\rho}(a_k))$.

Let us return to the case when $R(\mathbf{f}, a, z)$ is polynomial in z. Buchstaber and Rees's theory can be recovered as follows. The degree of $R(\mathbf{f}, a, z)$ in z equals $\mathbf{f}(1) = \chi$, hence $\chi = n \geqslant 0$. Therefore $\psi_k(\mathbf{f}, a) = 0$ for all $k \geqslant n+1$ and all $a \in A$. This is equivalent to the equations: $\mathbf{f}(1) = n \in \mathbb{N}$ and $\Phi_{n+1}(\mathbf{f}, a_1, \ldots, a_{n+1}) = 0$ for all a_i, which is precisely the definition of an n-homomorphism according to Buchstaber and Rees [2]. In this case $\operatorname{ber}(\mathbf{f}, a) = \psi_n(\mathbf{f}, a)$ (in particular, it is a polynomial function of a), therefore the function $\psi_n(\mathbf{f}, a)$ turns out to be multiplicative in a; hence its polarization $\Phi_n(\mathbf{f}, a_1, \ldots, a_n)/n!$ is an algebra homomorphism $S^nA \to B$. This gives a one-to-one correspondence between n-homomorphisms $A \to B$ and algebra homomorphisms $S^nA \to B$.

We define the *$p|q$-th symmetric power* $\operatorname{Sym}^{p|q} X$ of a topological space X as the identification space of the Cartesian product X^{p+q} with respect to the action of the group $S_p \times S_q$ and the relations

$$(x_1, \ldots, x_{p-1}, y, x_{p+1} \ldots, x_{p+q-1}, y) \sim (x_1, \ldots, x_{p-1}, z, x_{p+1} \ldots, x_{p+q-1}, z)\,.$$

An algebraic analog of the space $\operatorname{Sym}^{p|q} X$, for a commutative associative algebra with unit A, we define to be the subalgebra $S^{p|q}A := \mu^{-1}\left(S^{p-1}A \otimes S^{q-1}A\right)$ of the algebra $S^pA \otimes S^qA$ where $\mu\colon S^pA \otimes S^qA \to S^{p-1}A \otimes S^{q-1}A \otimes A$ is the multiplication of the last arguments. Example: for $A = \mathbb{C}[x]$, the algebra $S^{p|q}A$ will be the algebra of all polynomial invariants of $p|q$ by $p|q$ matrices (this is a non-trivial statement). There is a relation between algebra homomorphisms $S^{p|q}A \to B$ and $p|q$-homomorphisms $A \to B$. To each homomorphism $S^{p|q}A \to B$ canonically corresponds a $p|q$-homomorphism $A \to B$. (We have managed to establish the inverse in special cases.) Example. An element $[x_1, \ldots, x_{p+q}] \in \operatorname{Sym}^{p|q} X$ defines a $p|q$-homomorphism on $A = C(X)$:

$a \mapsto a(x_1) + \ldots + a(x_p) - \ldots - a(x_{p+q})$. In general, an integral linear combination of algebra homomorphisms of the form $\sum n_\alpha \mathbf{f}_\alpha$ where $n_\alpha \in \mathbb{Z}$ is a $p|q$-homomorphism with $p = \sum_{n_\alpha > 0} n_\alpha$, $q = -\sum_{n_\alpha < 0} n_\alpha$, and $\chi = \sum n_\alpha$.

The condition that $\mathbf{f}\colon A \to B$ is a $p|q$-homomorphism can be expressed by equations: $\mathbf{f}(1) = p - q$ and $|\psi_k(\mathbf{f}, a), \ldots, \psi_{k+q}(\mathbf{f}, a)|_{q+1} = 0$ for $k \geqslant p - q + 1$, where $|\psi_k(\mathbf{f}, a), \ldots, \psi_{k+q}(\mathbf{f}, a)|_{q+1}$ is a Hankel determinant, cf. [3]. This system of polynomial equations for 'coordinates' of the linear map $\mathbf{f}$ should, in particular, describe the image of $\operatorname{Sym}^{p|q} X$ in $C(X)^*$.

Our results may have an application to topological ramified coverings (cf. [4]). We thank V. M. Buchstaber for a fruitful discussion.

References

[1] I. M. Gel'fand and A. N. Kolmogorov. *Dokl. Akad. Nauk SSSR*, 22: 11–15, 1939.

[2] V. M. Buchstaber and E. G. Rees. *Russian Math. Surveys*, 59(1(355)):125–145, 2004.

[3] H. M. Khudaverdian and Th. Th. Voronov. *Lett. Math. Phys.*, 74(2):201–228, 2005.

[4] V. M. Buchstaber and E. G. Rees. Frobenius n-homomorphisms, transfers and branched coverings. *Math. Proc. Cambr. Phil. Soc.*, 2007 (to appear), `arXiv:math.RA/0608120`.

School of Mathematics, University of Manchester, Manchester, M60 1QD, United Kingdom

E-mail address: `theodore.voronov@manchester.ac.uk, khudian@manchester.ac.uk`