

# OPERATOR PENCIL PASSING THROUGH A GIVEN OPERATOR

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ABSTRACT. Let  $\Delta$  be a linear differential operator acting on the space of densities of a given weight  $\lambda_0$  on a manifold  $M$ . One can consider a pencil of operators  $\widehat{\Pi}(\Delta) = \{\Delta_\lambda\}$  passing through the operator  $\Delta$  such that any  $\Delta_\lambda$  is a linear differential operator acting on densities of weight  $\lambda$ . This pencil can be identified with a linear differential operator  $\widehat{\Delta}$  acting on the algebra of densities of all weights. The existence of an invariant scalar product in the algebra of densities implies a natural decomposition of operators, i.e. pencils of self-adjoint and anti-self-adjoint operators. We study lifting maps that are on one hand equivariant with respect to divergenceless vector fields, and, on the other hand, with values in self-adjoint or anti-self-adjoint operators. In particular we analyze the relation between these two concepts, and apply it to the study of  $\text{diff}(M)$ -equivariant liftings. Finally we briefly consider the case of liftings equivariant with respect to the algebra of projective transformations and describe all regular self-adjoint and anti-self-adjoint liftings.

## 1. INTRODUCTION

We say that  $\mathbf{s} = s(x)|Dx|^\lambda$  is a density of weight  $\lambda$  on a manifold  $M$  (which we assume to be orientable with a given orientation) if under a change of local coordinates it is multiplied by the  $\lambda$ -th power of the Jacobian of the coordinate transformation:

$$\mathbf{s} = s(x)|Dx|^\lambda = s(x(x')) \left( \det \left( \frac{\partial x}{\partial x'} \right) \right)^\lambda |Dx'|^\lambda. \quad (1)$$

Let  $\mathcal{F}_\lambda = \mathcal{F}_\lambda(M)$  denote the space of densities of weight  $\lambda$  on  $M$  for  $\lambda$  an arbitrary real number. Note that the space of functions on  $M$  is  $\mathcal{F}_0(M)$ , densities of weight  $\lambda = 0$ .

Differential operators on the space of densities of different weights have been under intensive study, see [8], [1], [2], [10], [3], [9], [4], [5] and the book [11] and citations therein. See also [6] and [7].

In these works the spaces  $\mathcal{D}_\lambda(M)$ , of linear differential operators defined on  $\mathcal{F}_\lambda(M)$  were studied. In particular in [8], [2] and [10], the problem of the existence of  $\text{diff}$ -equivariant maps between these spaces were studied. The spaces  $\mathcal{D}_\lambda(M)$  can be naturally considered as modules not only over group of diffeomorphisms but also over its subgroups such as projective or conformal transformations (provided the manifold is equipped with projective or conformal structure).

In the works [9], [3] the question concerning the existence of a quantisation map equivariant with respect to projective or conformal groups was studied. They constructed a unique full symbol calculus in each of the above cases, and used this to define the quantisation map. On the other hand authors of the works [6] and [7] whilst analysing Batalin-Vilkovisky

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geometry and  $\Delta$ -operators on densities of weight  $\frac{1}{2}$ , naturally came to the analysis of second order operators acting on the algebra of densities of all weights. Considering the canonical scalar product in this algebra, for the classification of second order operators on odd symplectic supermanifold, they come in particular to a self-adjoint pencil lifting of second order operators in the spaces  $\mathcal{D}_\lambda(M)$ . This result gave a clear geometrical picture to the isomorphisms between modules  $\mathcal{D}_\lambda^{(2)}(M)$  established in [2].

**1.1. Operator pencils and operators on algebra of densities.** One can consider pencils of differential operators (these shall also be referred to as operator pencils), i.e. a family  $\{\Delta_\lambda\}$  of operators depending on a parameter  $\lambda$ , such that for each  $\lambda$ ,  $\Delta_\lambda$  is a differential operator that acts on the space  $\mathcal{F}_\lambda$ . For a simple example of an operator pencil let  $x$  be the standard coordinate on the line  $\mathbb{R}$  then the formula

$$\{\Delta_\lambda\}: \quad \Delta_\lambda = A(\lambda)\frac{d^2}{dx^2} + B(\lambda)\frac{d}{dx} + C(\lambda), \quad \mathcal{F}_\lambda \rightarrow \mathcal{F}_\lambda, \quad (2)$$

where  $A(\lambda), B(\lambda), C(\lambda)$  are functions of  $\lambda$ , defines a pencil of second order operators on  $\mathbb{R}$ .

In this paper we shall study pencils of operators passing through a given operator which acts on densities of a given weight. For further considerations it will be useful to identify pencils of operators with operators acting on the algebra of *all* densities,  $\mathcal{F}(M) = \bigoplus_\lambda \mathcal{F}_\lambda(M)$ . (If  $\mathbf{s}_1 = s_1(x)|Dx|^{\lambda_1} \in \mathcal{F}_{\lambda_1}$  is a density of weight  $\lambda_1$  and  $\mathbf{s}_2 = s_2(x)|Dx|^{\lambda_2} \in \mathcal{F}_{\lambda_2}$  is a density of weight  $\lambda_2$  then their product is a density  $\mathbf{s}_1 \cdot \mathbf{s}_2 = s_1(x)s_2(x)|Dx|^{\lambda_1+\lambda_2}$  of weight  $\lambda_1 + \lambda_2$ . See in more detail [6].)

Consider the linear operator  $\widehat{\lambda}$  acting on the algebra  $\mathcal{F}(M)$  which multiplies a density by its weight:

$$\widehat{\lambda}\mathbf{s} = \lambda\mathbf{s} \quad \text{if } \mathbf{s} \in \mathcal{F}_\lambda: \quad \widehat{\lambda}(s(x)|Dx|^\lambda) = \lambda s(x)|Dx|^\lambda. \quad (3)$$

The operator  $\widehat{\lambda}$  (*weight operator*) is a first order linear differential operator on the algebra  $\mathcal{F}(M)$  as the Leibnitz rule is obeyed:

$$\widehat{\lambda}(\mathbf{s}_1 \cdot \mathbf{s}_2) = \widehat{\lambda}(\mathbf{s}_1) \cdot \mathbf{s}_2 + \mathbf{s}_1 \cdot (\widehat{\lambda}\mathbf{s}_2).$$

Respectively  $\widehat{\lambda}^n$  is an  $n$ -th order linear differential operator on  $\mathcal{F}(M)$ .

This observation allows us to consider operator pencils which depend polynomially on  $\lambda$  as differential operators on the algebra  $\mathcal{F}(M)$  of all densities. For example the operator pencil  $\{\Delta_\lambda\}: \Delta_\lambda = \lambda\frac{d^2}{dx^2} + (\lambda^2 + 1)\frac{d}{dx}$ , on  $\mathbb{R}$  (an operator pencil of the form (2) for  $A(\lambda) = \lambda, B(\lambda) = \lambda^2 + 1, C(\lambda) = 0$ ) can be considered as a third order operator

$$\widehat{\Delta} = \widehat{\lambda}\frac{\partial^2}{\partial x^2} + (\widehat{\lambda}^2 + 1)\frac{\partial}{\partial x}$$

on the algebra  $\mathcal{F}(\mathbb{R})$ . A general differential operator  $\widehat{\Delta}$  of order  $\leq n$  on the algebra  $\mathcal{F}(M)$  can be written locally as an expansion of the form

$$\widehat{\Delta} = L^{(n)} + \widehat{\lambda}L^{(n-1)} + \widehat{\lambda}^2L^{(n-2)} + \dots + \widehat{\lambda}^{n-1}L^{(1)} + \widehat{\lambda}^nL^{(0)}, \quad (4)$$

where  $\widehat{\lambda}$  is the weight operator (3), and the coefficients  $L^{(i)}$  in the expansion are usual differential operators of orders  $\leq i$  acting on densities. The corresponding operator pencil is

$$\{\Delta_\lambda\}: \Delta_\lambda = \widehat{\Delta}|_{\mathcal{F}_\lambda(M)} = L^{(n)} + \lambda L^{(n-1)} + \lambda^2 L^{(n-2)} + \dots + \lambda^{n-1} L^{(1)} + \lambda^n L^{(0)}. \quad (5)$$

It is useful to consider vertical operators and vertical maps on the algebra of densities:

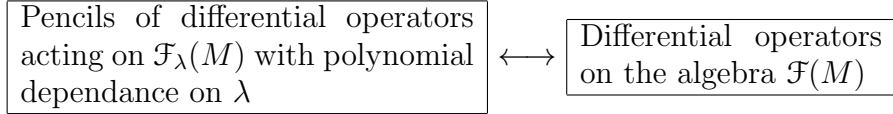
**Definition 1.** An operator  $\widehat{\Delta}$  acting on  $\mathcal{F}(M)$  is called vertical if for an arbitrary function  $f$  and a density  $\mathbf{s}$

$$\widehat{\Delta}(f\mathbf{s}) = f\widehat{\Delta}(\mathbf{s}). \quad (6)$$

If  $\widehat{\Delta}$  is vertical operator, then  $\widehat{\Delta} = \sum \widehat{\lambda}^k c_k(x)$ , and the restriction of it to any space  $\mathcal{F}_\lambda$  is an operator of order 0, i.e. multiplication by a function.

Similarly we shall call a map of operators vertical if its image lies in the space of vertical operators.

Later by default we shall only consider operator pencils depending polynomially on the weight  $\lambda$



We shall continuously make the identification above throughout the text, namely operator pencils with their corresponding operators on  $\mathcal{F}(M)$ .

We shall formulate now explicitly what we mean when we say that we “draw a pencil” through a given operator. Denote by  $\mathcal{D}_\lambda^{(n)}(M)$  the space of linear differential operators of order  $\leq n$  acting on the space  $\mathcal{F}_\lambda(M)$  and by  $\widehat{\mathcal{D}}^{(n)}(M)$  the space of linear differential operators of order  $\leq n$  acting on algebra  $\mathcal{F}(M)$ <sup>1</sup>. We also consider the spaces  $\mathcal{D}_\lambda(M) = \cup_n \mathcal{D}_\lambda^{(n)}(M)$  of linear differential operators of all orders acting on  $\mathcal{F}_\lambda$  and respectively the space  $\widehat{\mathcal{D}}(M) = \cup_n \widehat{\mathcal{D}}^{(n)}(M)$  of linear differential operators of all orders acting on  $\mathcal{F}(M)$ .

Recall that an arbitrary linear differential operator  $\widehat{\Delta}$  on  $\mathcal{F}(M)$  may be identified with an operator pencil  $\{\Delta_\lambda\}$ ,  $\widehat{\Delta}|_{\widehat{\lambda}=\lambda} = \Delta_\lambda$  ( $\widehat{\Delta}|_{\widehat{\lambda}=\lambda} = \widehat{\Delta}|_{\mathcal{F}_\lambda(M)}$ ) polynomially depending on  $\lambda$ , thus in particular an operator  $\widehat{\Delta}$  of order  $\leq n$  acting on  $\mathcal{F}(M)$ , ( $\widehat{\Delta} \in \widehat{\mathcal{D}}^{(n)}(M)$ ) can be considered as an operator pencil, which is a polynomial in  $\lambda$ , of order  $\leq n$  such that terms which have order  $k$  over the weight operator  $\widehat{\lambda}$  ( $k \leq n$ ) possess derivatives over the coordinates  $x^i$  of order  $\leq n - k$  (see equations (4) and (5)).

**Definition 2.** We say that the operator  $\widehat{\Delta}$  on the algebra of densities is a pencil lifting of the operator  $\Delta$ , (or just a lifting of  $\Delta$ ) if the restriction of  $\widehat{\Delta}$  to the space  $\mathcal{F}_\lambda(M)$  is the operator  $\Delta$  itself:

$$\widehat{\Delta}|_{\widehat{\lambda}=\lambda} = \Delta.$$

One can say that the operator pencil, which is identified with the operator  $\widehat{\Delta}$ , passes through the operator  $\Delta$ .

We consider linear maps  $\widehat{\Pi}$  from differential operators on densities of an arbitrary but given weight  $\lambda$ , to operators on the algebra of all densities. We say that a map  $\widehat{\Pi}$ , defined on operators acting on the space  $\mathcal{F}_\lambda(M)$ , is a pencil lifting map (or just shortly lifting map) if the operator  $\widehat{\Delta} = \widehat{\Pi}(\Delta)$  is a pencil lifting for any operator  $\Delta$ .

<sup>1</sup>An operator  $\widehat{\Delta} \in \mathcal{D}^{(n)}(M)$  has an order  $\leq n$  if for an arbitrary density  $\mathbf{s}$  the commutator with the multiplication operator  $[\widehat{\Delta}, \mathbf{s}] = \mathbf{s} \circ \widehat{\Delta} - \widehat{\Delta} \circ \mathbf{s}$  is an operator of order  $\leq n - 1$ . Respectively an operator  $\Delta \in \mathcal{D}_\lambda^{(n)}$ ,  $\Delta: \mathcal{F}_\lambda \rightarrow \mathcal{F}_\lambda$  has order  $\leq n$  if the operator  $[\Delta, f] = f \circ \Delta - \Delta \circ f$  has order  $\leq n - 1$  for an arbitrary function  $f$ .

**Remark 1.** In this article we will only consider operators of weight 0, i.e. operators which do not alter the weight of densities. (An operator  $\Delta$  has weight  $\delta$  if  $[\widehat{\lambda}, \Delta] = \delta\Delta$ .) In fact there are many interesting phenomena associated to operators of non-zero weights (see for example the book [11] and citations therein). In our analysis these results crop up as well and we will consider them later.

**Definition 3.** Pick an arbitrary  $\lambda_0 \in \mathbb{R}$  and arbitrary  $n$ . Let  $\widehat{\Pi}$  be a pencil lifting map defined on the space  $\mathcal{D}_{\lambda_0}^{(n)}(M)$ :

$$\widehat{\Pi}: \mathcal{D}_{\lambda_0}^{(n)}(M) \rightarrow \widehat{\mathcal{D}}(M), \quad \widehat{\Pi}|_{\mathcal{F}_{\lambda_0}(M)} = \mathbf{id}.$$

We say that the lifting map  $\widehat{\Pi}$  is a *regular* lifting of the space  $\mathcal{D}_{\lambda_0}^{(n)}(M)$  if it takes values in operators of order  $\leq n$ :

$$\forall \Delta \in \mathcal{D}_{\lambda_0}^{(n)}(M), \quad \widehat{\Pi}(\Delta) \in \widehat{\mathcal{D}}^{(n)}(M).$$

We say that  $\widehat{\Pi}$  is a *strictly regular* lifting map on  $\mathcal{D}_{\lambda_0}^{(n)}(M)$  if its restriction on every subspace  $\mathcal{D}_{\lambda_0}^{(k)}(M) \subseteq \mathcal{D}_{\lambda_0}^{(n)}(M)$ , ( $k \leq n$ ) is a regular lifting map:

$$\forall k \leq n, \forall \Delta \in \mathcal{D}_{\lambda_0}^{(k)}(M), \quad \widehat{\Pi}(\Delta) \in \widehat{\mathcal{D}}^{(k)}(M).$$

In other words for a given  $n$ , regular liftings map operators of order  $\leq n$  to operators of order  $\leq n$ , and strictly regular liftings do not increase the order of any operator  $\Delta$  which has order less or equal than  $n$ .

**Example 1.1.** Let  $\mathcal{D}_0^{(2)}(\mathbb{R}^n)$  be the space of second order operators on functions on Cartesian space  $\mathbb{R}^n$ . Consider the map

$$\widehat{\Pi}(\Delta) = \Delta + \widehat{\lambda}\Delta_1 = S^{ij}(x)\partial_i\partial_j + A^i(x)\partial_i + F(x) + \widehat{\lambda}(aS^{ij}(x)\partial_i\partial_j + b(\partial_k S^{ki}(x) + A^i(x))\partial_i),$$

with values in operator pencils, operators on the space of all densities on  $\mathbb{R}^n$ , where  $a$  and  $b$  are arbitrary parameters. This is a lifting map since  $\widehat{\Delta} = \Pi(\Delta)|_{\lambda=0} = \Delta$  and assigns to the second order operator  $\Delta = S^{ij}\partial_i\partial_j + A^i\partial_i + F$  the pencil  $\{\Delta_\lambda\}$  of second order operators,  $\Delta + \lambda\Delta_1$  passing through the operator  $\Delta$ . On the other hand this pencil defines a *third* order operator on the algebra of densities in the case if  $a \neq 0$  and  $S^{ij} \neq 0$ , since the operator  $\widehat{\lambda}\Delta_1 = a\widehat{\lambda}S^{ij}\partial_i\partial_j + \dots$  acting on  $\mathcal{F}(\mathbb{R}^n)$  has order three. We see that the lifting map  $\Pi(\Delta)$  is a regular map if and only if  $a = 0$ . This regular map is strictly regular if and only if  $b = 0$  also, since in the case  $\Delta$  is a first order operator and  $b \neq 0$ , the operator  $\Pi(\Delta) = \widehat{\lambda}bA^i\partial_i$  is a second order operator. (We will give another more geometrical examples later in the text.)

In what follows we consider liftings maps that are equivariant with respect to some subgroup of the group of diffeomorphisms or with respect to some subalgebra of the algebra of vector fields (infinitesimal diffeomorphisms).

A lifting map,  $\Delta \mapsto \widehat{\Pi}(\Delta)$ , is equivariant with respect to the diffeomorphism  $\varphi$  if  $\varphi^*(\widehat{\Pi}(\Delta)) = \widehat{\Pi}(\varphi^*\Delta)$  for an arbitrary operator  $\Delta$ . Respectively, a lifting  $\Delta \mapsto \widehat{\Pi}(\Delta)$  is equivariant with respect to the vector field  $\mathbf{X}$  on  $M$  (an infinitesimal diffeomorphism) if

$$\text{ad}_{\mathbf{X}}\widehat{\Pi}(\Delta) = \widehat{\Pi}(\text{ad}_{\mathbf{X}}\Delta),$$

for an arbitrary operator  $\Delta$ , where we denote by  $\text{ad}_{\mathbf{X}}$  the action of  $\mathbf{X}$  on an operator  $\Delta$ : for an arbitrary density  $\Psi$

$$\text{ad}_{\mathbf{X}}\Delta(\Psi) = \mathcal{L}_{\mathbf{X}}(\Delta(\Psi)) - \Delta(\mathcal{L}_{\mathbf{X}}(\Psi)),$$

where  $\mathcal{L}_{\mathbf{X}}$  is the Lie derivative with respect to vector field  $\mathbf{X}$ .

**Definition 4.** Let  $G \subseteq \text{Diff}(M)$  be a subgroup and let  $\mathcal{G} = \mathfrak{G}(G)$  be the corresponding Lie subalgebra of vector fields. A pencil lifting map  $\widehat{\Pi}$  is  $G$ -equivariant if it is equivariant with respect to all  $\varphi \in G$ . Respectively a lifting map  $\widehat{\Pi}$  is  $\mathcal{G}$ -equivariant if it is equivariant with respect to an arbitrary vector field  $\mathbf{X} \in \mathcal{G}$ . (We will also refer to such maps as  $G$ -liftings respectively  $\mathcal{G}$ -liftings.)

In this article we primarily consider  $\text{SDiff}_{\rho}(M)$ , that is the group of diffeomorphisms that preserve a volume  $\rho$  on  $M$ , and the corresponding algebra,  $\text{sdiff}_{\rho}(M)$ , of divergenceless vector fields:

$$\mathbf{X} \in \text{sdiff}_{\rho}: \quad \text{div}_{\rho} \mathbf{X} = \rho^{-1} \mathcal{L}_{\mathbf{X}} \rho = \partial_i X^i(x) + X^i(x) \partial_i \log \rho(x) = 0, \quad (\rho = \rho(x)|Dx|). \quad (7)$$

We shall also discuss equivariance with respect to the full group of diffeomorphisms, and in the final section we shall consider projective transformations.

Technically it is more convenient to consider invariance with respect to a Lie algebra rather than the group. We will mostly consider liftings equivariant with respect to a Lie algebra.

On  $\mathcal{F}(M)$  one can consider the canonical scalar product,  $\langle \cdot, \cdot \rangle$ , defined by the following formula: for  $\mathbf{s}_1 = s_1(x)|Dx|^{\lambda_1}$  and  $\mathbf{s}_2 = s_2(x)|Dx|^{\lambda_2}$  then

$$\langle \mathbf{s}_1, \mathbf{s}_2 \rangle = \begin{cases} \int_M s_1(x)s_2(x)|Dx|, & \text{if } \lambda_1 + \lambda_2 = 1, \\ 0 & \text{if } \lambda_1 + \lambda_2 \neq 1. \end{cases} \quad (8)$$

This implies that a linear differential operator  $\widehat{\Delta}$  acting on the algebra  $\mathcal{F}(M)$  has an adjoint operator  $\widehat{\Delta}^*$ :

$$\langle \Delta \mathbf{s}_1, \mathbf{s}_2 \rangle = \langle \mathbf{s}_1, \Delta^* \mathbf{s}_2 \rangle \quad \left( \widehat{\Delta}^* \right) |_{\widehat{\lambda}=\lambda} = \left( \widehat{\Delta} \right) |_{\widehat{\lambda}=1-\lambda}^* . \quad (9)$$

One can see that  $\widehat{\lambda}^* = 1 - \widehat{\lambda}$ ,  $(\partial/\partial x^i)^* = -(\partial/\partial x^i)$ . (For details see [6] or [7].) We thus come to the notion of a self-adjoint (anti-self-adjoint) operator on the algebra of densities. These constructions, which were introduced in [6] for analysis of second order operators, will be in active use in our considerations. In particular we can consider self-adjoint and anti-self-adjoint lifting maps, where a lifting map,  $\widehat{\Pi}$ , is called self-adjoint (anti-self-adjoint) if for an arbitrary operator  $\Delta$  the operator  $\widehat{\Pi}(\Delta)$  is a self-adjoint (anti-self-adjoint) operator on the algebra of densities.

**Remark 2.** If  $\Delta = \widehat{\lambda}^r S^{i_1 \dots i_{n-r}} \partial_{i_1} \dots \partial_{i_{n-r}} + \dots$  is an  $n$ -th order operator such that the tensor  $S^{i_1 \dots i_{n-r}}$  does not vanish identically, then

$$\begin{aligned} \Delta^* &= (\widehat{\lambda}^*)^r (-1)^{n-r} S^{i_1 \dots i_{n-r}} \partial_{i_1} \dots \partial_{i_{n-r}} + \dots = \\ &= (1 - \widehat{\lambda})^r (-1)^{n-r} S^{i_1 \dots i_{n-r}} \partial_{i_1} \dots \partial_{i_{n-r}} + \dots = (-1)^n \widehat{\lambda}^r S^{i_1 \dots i_{n-r}} \partial_{i_1} \dots \partial_{i_{n-r}} + \dots, \end{aligned}$$

i.e. the operator  $\Delta - (-1)^n \Delta^*$  is an operator of the order  $\leq n - 1$ .

In particular it follows from this fact that if we consider liftings of operators defined on the spaces  $\mathcal{D}_{\lambda_0}^{(n)}$  (operators of the order  $\leq n$  on densities of weight  $\lambda_0$ ), then self-adjoint pencil liftings have to be considered if  $n$  is an even number, and anti-self-adjoint pencil liftings if  $n$  is an odd number.

This article contains the following:

In section 2 we recall constructions of canonical pencils on the spaces of first order and second order operators. Pencil liftings for first order operators is just a reformulation of standard exercises in differential geometry concerning Lie derivatives of functions and densities. For second order operators we briefly recall the constructions suggested in [6], where a canonical self-adjoint pencil of second order operators was constructed. It turns out that for every second order operator  $\Delta \in \mathcal{D}_\lambda(M)$  ( $\lambda \neq 0, 1/2, 1$ ) there exists a unique such a pencil. This pencil provides us with a regular  $\text{diff}(M)$ -equivariant pencil lifting map on  $\mathcal{D}_\lambda^{(2)}(M)$ . Restricting this pencil for different values of  $\lambda'$  leads to isomorphisms of  $\text{diff}$ -modules  $\mathcal{D}_\lambda^{(2)}$  and  $\mathcal{D}_{\lambda'}^{(2)}$  ( $\lambda, \lambda' \neq 0, 1/2, 1$ ). These are just the isomorphisms established in the work [2] of Duval and Ovsienko. The canonical self-adjoint pencil reveals not only the geometrical meaning of their isomorphisms but also indicates the special role of self-adjointness for maps between spaces of operators.

What about maps between spaces of higher order operators? Results of the works [8] and [10] imply sort of no go theorem that there do not exist  $\text{diff}(M)$ -equivariant liftings on  $\mathcal{D}_\lambda^{(n)}(M)$ , if manifold  $M$  has dimension greater than 1. We shall attempt to analyze the geometrical reasons behind this. In section 3 and 4 we consider a smaller algebra of divergenceless vector fields,  $\text{sdiff}(M)$ , corresponding to the group  $\text{SDiff}(M)$  of diffeomorphisms preserving a volume form. On one hand the volume form identifies all the spaces  $\mathcal{F}_\lambda(M)$  and hence the spaces of operators  $\mathcal{D}_\lambda(M)$ . On the other hand any candidate for  $\text{diff}(M)$ -equivariant map has to pass the test of being at least  $\text{sdiff}(M)$ -equivariant. We describe all regular  $\text{sdiff}$ -equivariant pencil liftings and (anti)-self-adjoint regular  $\text{sdiff}$ -pencil liftings if manifold  $M$  has dimension greater than 2. It turns out that only self-adjoint  $\text{sdiff}_\rho(M)$ -liftings have a chance to be  $\text{diff}(M)$ -equivariant. Namely we see how a pencil lifting depends on a volume form, and come to the conclusion that if  $\text{sdiff}_\rho$ -lifting map does not depend on a volume form  $\rho$  (i.e. it is  $\text{diff}(M)$ -lifting) at least in a vicinity of a given operator then the map is (anti)-self-adjoint in a vicinity of this point (up to a vertical operator) (see corollary 1 below).

In the section 5 we suggest Taylor series expansion of operators on algebra of densities with respect to vertical operators. This Taylor series expansion may be useful for different tasks. In particular we use this expansion to describe all (anti)-self-adjoint liftings of an arbitrary operator acting on densities.

Finally in section 6 we briefly consider regular pencil liftings which are equivariant with respect to the algebra of projective transformations. We write down the formula for all regular liftings equivariant with respect to infinitesimal projective transformations of  $\mathbb{R}^d$  and calculate those that are (anti)-self-adjoint amongst them.

**Remark 3.** We only consider linear maps on operators that are differential polynomials. These maps are non-other than those that are local by Peetre's theorem [12]. (In fact in the case of equivariant maps it can be shown that for suitable algebras of vector fields this is automatically satisfied (details can be found in [8] or [9].))

In this article by default the manifold  $M$  under consideration is oriented and therefore the transformation laws (1) are well defined for *arbitrary*  $\lambda$ . For the adjoint of an operator to be well defined we suppose that  $M$  is compact, or it is an open domain in  $\mathbb{R}^m$ . In the latter case we assume that the functions are rapidly decreasing at infinity.

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## 2. CANONICAL LIFTINGS OF FIRST AND SECOND ORDER OPERATORS

In this section we consider canonical constructions that give us liftings of first and second order operators which are  $\text{Diff}(M)$ -equivariant. In subsequent sections we will see that all  $\text{Diff}(M)$ -liftings that are defined on all operators of a given weight and order are contained in these examples.

**2.1. Lifting of first order operators.** A vector field  $\mathbf{X}$  on  $M$  defines a Lie derivative  $\mathcal{L}_{\mathbf{X}}^\lambda$ , a first order operator on the space  $\mathcal{F}_\lambda(M)$  of densities of weight  $\lambda$ . If in local coordinates  $(x^i)$ ,  $\mathbf{X} = X^i(x)\partial_i$  ( $\partial_i = \frac{\partial}{\partial x^i}$ ) then

$$\mathcal{L}_{\mathbf{X}}^\lambda(\mathbf{s}(x)) = \mathcal{L}_{\mathbf{X}}^\lambda(s(x)|Dx|^\lambda) = (X^i(x)\partial_i s(x) + \lambda\partial_i X^i(x)s(x)) |Dx|^\lambda.$$

A pencil of Lie derivatives  $\{\mathcal{L}_{\mathbf{X}}^\lambda\}$  can be identified with the Lie derivative  $\widehat{\mathcal{L}}_{\mathbf{X}}$ , a first order operator on  $\mathcal{F}(M)$ :

$$\widehat{\mathcal{L}}_{\mathbf{X}} = X^i(x)\partial_i s(x) + \widehat{\lambda}\partial_i X^i(x), \quad \left(\widehat{\mathcal{L}}_{\mathbf{X}}\right) \Big|_{\widehat{\lambda}=\lambda} = \mathcal{L}_{\mathbf{X}}^\lambda. \quad (10)$$

This is an anti-self-adjoint operator:

$$\left(\widehat{\mathcal{L}}_{\mathbf{X}}\right)^* = \left(X^i(x)\partial_i + \widehat{\lambda}\partial_i X^i\right)^* = -X^i(x)\partial_i - \partial_i X^i(x) + \widehat{\lambda}^*\partial_i X^i(x) = -\widehat{\mathcal{L}}_{\mathbf{X}}$$

since  $\widehat{\lambda}^* = 1 - \widehat{\lambda}$ . In other words the operator  $(\mathcal{L}_{\mathbf{X}}^\lambda)^*$  acting on densities of weight  $(1 - \lambda)$  is equal to  $-\mathcal{L}_{\mathbf{X}}^{1-\lambda}$ .

Pick any  $\lambda_0$  and consider a space  $\mathcal{D}_{\lambda_0}^{(1)}(M)$  of first order operators acting on densities of weight  $\lambda_0$ . For an arbitrary operator  $\Delta \in \mathcal{D}_{\lambda_0}^{(1)}(M)$  consider the vector field  $\mathbf{A}$ , the principal symbol of the operator  $\Delta$ :  $(\mathbf{A}f)\mathbf{s} = \Delta(f\mathbf{s}) - f\Delta(\mathbf{s})$ , where  $f$  is an arbitrary function and  $\mathbf{s}$  is an arbitrary density of weight  $\lambda_0$  on  $M$ . The difference between the operator  $\Delta$  and the Lie derivative  $\mathcal{L}_{\mathbf{A}}^{\lambda_0}$  is a zeroth order operator. We thus come to the conclusion that an arbitrary first order operator  $\Delta$  can be canonically decomposed into the sum of a Lie derivative and a scalar function:

$$\begin{aligned} \Delta &= A^i(x)\partial_i + B(x) = \mathcal{L}_{\mathbf{A}}^{(\lambda_0)} + S(x), \\ \text{where } S(x) &= \Delta - \mathcal{L}_{\mathbf{A}}^{(\lambda_0)} = B(x) - \lambda_0\partial_i A^i(x). \end{aligned} \quad (11)$$

Using this decomposition we come to a canonical pencil lifting map defined on the space  $\mathcal{D}_{\lambda_0}^{(1)}(M)$ :

$$\widehat{\Pi}(\Delta) = \widehat{\mathcal{L}}_{\mathbf{A}} + S(x), \quad \text{if } \Delta = \mathcal{L}_{\mathbf{A}}^{\lambda_0} + S(x),$$

where  $\widehat{\mathcal{L}}_{\mathbf{A}}$  is the Lie derivative (10). In local coordinates

$$\widehat{\Pi}(\Delta) = \widehat{\Pi}(A^i \partial_i + B(x)) = \underbrace{A^i(x) \partial_i + \widehat{\lambda} \partial_i A^i(x)}_{\widehat{\mathcal{L}}_{\mathbf{A}}} + \underbrace{B(x) - \lambda_0 \partial_i A^i(x)}_{\text{scalar function}}. \quad (12)$$

This is a pencil lifting map,  $\widehat{\Pi}(\Delta)|_{\widehat{\lambda}=\lambda_0} = \Delta$ , it is strictly regular and it is obviously a  $\text{Diff}(M)$ -lifting since the map is canonical, it does not depend on a choice of local coordinates.

One can twist this lifting to get a family of regular pencil liftings:  $\Pi(\Delta) = \widehat{\mathcal{L}}_{\mathbf{A}} + C(\widehat{\lambda})S(x)$ , where  $C(\widehat{\lambda})$  is a polynomial of order  $\leq 1$  such that it obeys the condition  $C(\widehat{\lambda})|_{\widehat{\lambda}=\lambda_0} = 1$ , i.e.  $C(\widehat{\lambda}) = 1 + c(\widehat{\lambda} - \lambda_0)$ , where  $c$  is an arbitrary constant. We come to the following regular pencil liftings of  $\mathcal{D}_{\lambda_0}^{(1)}(M)$ :

$$\widehat{\Pi}_c(\Delta) = \widehat{\mathcal{L}}_{\mathbf{A}} + \left(1 + c(\widehat{\lambda} - \lambda_0)\right) S(x) =$$

$$A^i(x) \partial_i + \widehat{\lambda} \partial_i A^i(x) + \left(1 + c(\widehat{\lambda} - \lambda_0)\right) (B(x) - \lambda_0 \partial_i A^i(x)), \quad c \in \mathbb{R}. \quad (13)$$

This formula presents a one-parametric family, an affine line, of  $\text{Diff}(M)$ -equivariant regular pencil liftings. This affine line possesses the distinguished point  $c = 0$ , the unique point that corresponds to a strictly regular pencil (12). All other points on the line are not strictly regular liftings (if  $\mathbf{A} \equiv 0$  and  $B(x) \neq 0$ , then  $\Delta$  is a zeroth order operator while  $\Pi(\Delta)$  is a first order operator).

In the case if our original weight  $\lambda_0 \neq \frac{1}{2}$  the affine line of liftings maps possesses another distinguished point: if  $c = \frac{2}{2\lambda_0 - 1}$  then the lifting (13) becomes

$$\widehat{\Pi}(\Delta) = \widehat{\mathcal{L}}_{\mathbf{A}} + \frac{2\widehat{\lambda} - 1}{2\lambda_0 - 1} S(x) = A^i(x) \partial_i + \widehat{\lambda} \partial_i A^i(x) + \frac{2\widehat{\lambda} - 1}{2\lambda_0 - 1} (B(x) - \lambda_0 \partial_i A^i(x)). \quad (14)$$

This is an anti-self-adjoint lifting:  $\left(\widehat{\Pi}(\Delta)\right)^* = -\widehat{\Pi}(\Delta)$  since  $\widehat{\mathcal{L}}_{\mathbf{A}}^* = -\widehat{\mathcal{L}}_{\mathbf{A}}$  and  $(2\widehat{\lambda} - 1)^* = (2(1 - \widehat{\lambda}) - 1) = -(2\widehat{\lambda} - 1)$ .

**Remark 4.** Note that for an arbitrary first order operator  $\widehat{\Delta} = M^i(x) \partial_i + \widehat{\lambda} N(x) + P(x)$  the condition of anti-self-adjointness,  $\Delta^* = -\Delta$ , is equivalent to the condition that  $N = \partial_i M^i(x) - 2P(x)$ , i.e.

$$\Delta = M^i(x) \partial_i + \widehat{\lambda} \partial_i M(x) - (2\widehat{\lambda} - 1)P(x) = \widehat{\mathcal{L}}_{\mathbf{M}} - (2\widehat{\lambda} - 1)P(x).$$

Comparing with equation (14) we see that in the case  $\lambda_0 \neq \frac{1}{2}$  there is a unique anti-self-adjoint regular lifting map defined on the space  $\mathcal{D}_{\lambda_0}^{(1)}(M)$ .



**2.2. Liftings of second order operators.** We described liftings of first order operators using Lie derivatives (10), which are anti-self-adjoint first order operators. To study canonical liftings of second order operators it is useful to recall the description of self-adjoint second order operators (for details see [6] or [7]). Let  $\widehat{\Delta}$  be an arbitrary second order operator on  $\mathcal{F}(M)$ :

$$\widehat{\Delta} = \underbrace{S^{ij}(x)\partial_i\partial_j + \widehat{\lambda}B^i(x)\partial_i + \widehat{\lambda}^2C(x)}_{\text{second order derivatives}} + \underbrace{D^i(x)\partial_i + \widehat{\lambda}E(x)}_{\text{first order derivatives}} + F(x).$$

The adjoint of this operator has the form:

$$\Delta^* = S^{ij}(x)\partial_i\partial_j + 2\partial_i S^{ij}(x) + \partial_i\partial_j S^{ij}(x) + (\widehat{\lambda} - 1)(B^i(x)\partial_i + \partial_i B^i(x)) + (\widehat{\lambda} - 1)^2 C(x) - (\widehat{\lambda} - 1)E(x) - D^i\partial_i - \partial_i D^i(x) + F(x).$$

We see that the condition  $\Delta^* = \Delta$  is equivalent to

$$\widehat{\Delta} = S^{ij}(x)\partial_i\partial_j + \partial_j S^{ji}(x)\partial_i + (2\widehat{\lambda} - 1)\gamma^i(x)\partial_i + \widehat{\lambda}\partial_i\gamma^i(x) + \widehat{\lambda}(\widehat{\lambda} - 1)\theta(x) + F(x). \quad (15)$$

Here we denote  $\gamma^i(x) = B^i(x)/2$  and  $\theta(x) = C(x)$ . The coefficients of the operator (15) have the following geometrical meaning:

- $S^{ij}$  is a symmetric contravariant tensor field,
- $\gamma^i$  is an upper connection (see appendix A),
- $\theta(x)$  is a Brans-Dicke function, (see appendix A),
- $F(x)$  is a scalar function. Usually we enforce the normalisation condition

$$F(x) = \widehat{\Delta}(1) = 0. \quad (16)$$

Pick an arbitrary  $\lambda_0$  and let  $\Delta = S^{ij}(x)\partial_i\partial_j + T^i(x)\partial_i + R(x) \in \mathcal{D}_{\lambda_0}^{(2)}(M)$ . The self-adjoint operator  $\widehat{\Delta}$  defined by equation (15) is a lifting of the operator  $\Delta$ , i.e.  $\widehat{\Delta}|_{\widehat{\lambda}=\lambda_0} = \Delta$ , if the following conditions hold:

$$T^i(x) = \partial_j S^{ji}(x) + (2\lambda_0 - 1)\gamma^i(x), \quad R(x) = \lambda_0\partial_i\gamma^i(x) + \lambda_0(\lambda_0 - 1)\theta(x) + F(x). \quad (17)$$

We come to the statement that if  $\lambda_0 \neq 0, \frac{1}{2}, 1$  then for every operator  $\Delta \in \mathcal{D}_{\lambda_0}^{(2)}(M)$ , there exists a unique second order self-adjoint operator pencil,  $\widehat{\Delta}$ , which obeys the normalisation condition (16), and passes through this operator  $\Delta$ . This pencil is defined by equations (15), (17) and (16). In other words an arbitrary second order operator  $\Delta \in \mathcal{D}_{\lambda_0}^{(2)}(M)$  ( $\lambda_0 \neq 0, \frac{1}{2}, 1$ ) uniquely defines the geometrical data  $(S^{ik}, \gamma^i, \theta)$  (a symmetric contravariant tensor field of rank 2, an upper connection and a Brans-Dicke function), and these geometrical objects uniquely define the self-adjoint second order operator (15) on  $\mathcal{D}_{\lambda_0}^{(2)}(M)$  if we enforce normalisation condition (16). (For more detail see [6], [7].) This canonical map does not depend on a choice of local coordinates. Thus we have defined a regular, self-adjoint,  $\text{diff}(M)$ -equivariant pencil lifting map on the space  $\mathcal{D}_{\lambda_0}^{(2)}(M)$  for  $\lambda_0 \neq 0, \frac{1}{2}, 1$ .

Do there exist any other regular  $\text{diff}(M)$ -liftings? Uniqueness of this pencil map on  $\mathcal{D}_{\lambda_0}^{(2)}(M)$  follows from a theorem proved by Duval and Ovsienko in [2]. In the following sections we will give an alternative proof of this result when analysing  $\text{diff}(M)$ -liftings for higher order operators (see section 4.3).

3. Diff ( $M$ ) AND SDiff ( $M$ )-EQUIVARIANT LIFTINGS

In this section we consider SDiff ( $M$ )-equivariant regular lifting maps, which we will then use for analysing regular Diff ( $M$ )-lifting maps.

Let  $M$  be an orientable compact manifold provided with an orientation. We say that  $M$  has a volume form structure if a volume form  $\boldsymbol{\rho}$  is defined on  $M$ . Consider the group SDiff  $\boldsymbol{\rho}$  of orientation preserving diffeomorphisms which preserve volume form  $\boldsymbol{\rho}$ , and its corresponding Lie algebra sdiff  $\boldsymbol{\rho}(M)$  of divergence-less vector fields (see condition (7)).

On a manifold with a volume form  $\boldsymbol{\rho} = \rho(x)|Dx|$  an arbitrary density  $\mathbf{s}_1 = s_1|Dx|_1^\lambda$  of weight  $\lambda_1$  can be canonically identified with a density  $\mathbf{s}_2 = \mathbf{s}_1\boldsymbol{\rho}^{\lambda_2-\lambda_1} = s_1(x)\rho(x)^{\lambda_2-\lambda_1}|Dx|^{\lambda_2}$  of weight  $\lambda_2$ . Thus the two spaces  $\mathcal{F}_{\lambda_1}(M)$  and  $\mathcal{F}_{\lambda_2}(M)$  of densities of weights  $\lambda_1$  and  $\lambda_2$  are naturally identified. This implies canonical isomorphisms between differential operators on the spaces of densities:

$$P_{\lambda_2, \lambda_1}^\rho: \mathcal{D}_{\lambda_1}(M) \rightarrow \mathcal{D}_{\lambda_2}(M), \quad P_{\lambda_2, \lambda_1}^\rho(\Delta_{\lambda_1}) = \boldsymbol{\rho}^{\lambda_2-\lambda_1} \circ \Delta_{\lambda_1} \circ \boldsymbol{\rho}^{\lambda_1-\lambda_2}. \quad (18)$$

The isomorphisms  $P_{\lambda_1, \lambda_2}^\rho$  define lifting maps on the spaces  $\mathcal{D}_\lambda(M)$ . Namely choose an arbitrary  $\lambda_0$  and assign to every operator  $\Delta_{\lambda_0}$  on densities of weight  $\lambda_0$ , a pencil of operators  $\{\Delta_\lambda\}$  such that  $\Delta_\lambda = P_{\lambda, \lambda_0}^\rho(\Delta_{\lambda_0})$ . This pencil can be identified with an operator on the algebra of densities:

$$\widehat{\Delta} = \widehat{P}_{\lambda_0}^\rho(\Delta_{\lambda_0}): \widehat{\Delta}|_{\widehat{\lambda}=\lambda} = P_{\lambda, \lambda_0}^\rho(\Delta_{\lambda_0}). \quad (19)$$

This formula defines a natural lifting  $\widehat{P}_{\lambda_0}^\rho$  on the space  $\mathcal{D}_{\lambda_0}(M)$ . In local coordinates,  $(x^i)$ , it has the following appearance:

$$\begin{aligned} \Delta_{\lambda_0} = \sum L^{i_1 \dots i_k}(x) \partial_{i_1} \dots \partial_{i_k} &\mapsto \widehat{\Delta} = \widehat{P}_{\lambda_0}^\rho(\Delta_{\lambda_0}) = \boldsymbol{\rho}^{\widehat{\lambda}-\lambda_0} \circ \Delta_{\lambda_0} \circ \boldsymbol{\rho}^{\lambda_0-\widehat{\lambda}} = \\ &\rho^{\widehat{\lambda}-\lambda_0}(x) \circ \left( \sum L^{i_1 \dots i_k}(x) \partial_{i_1} \dots \partial_{i_k} \right) \circ \rho^{\lambda_0-\widehat{\lambda}}(x) = \\ &\sum L^{i_1 \dots i_k}(x) \left( \partial_{i_1} + (\widehat{\lambda} - \lambda_0) \Gamma_{i_1}(x) \right) \dots \left( \partial_{i_k} + (\widehat{\lambda} - \lambda_0) \Gamma_{i_k}(x) \right). \end{aligned} \quad (20)$$

Here  $\boldsymbol{\rho} = \rho(x)|Dx|$  is the volume form in coordinates  $x^i$  and  $\Gamma_i = -\partial_i(\log \rho(x))$  is a flat connection on densities of weight  $\lambda = 1$  corresponding to the volume form  $\boldsymbol{\rho} = \rho(x)|Dx|$ . In brief we come to the canonical pencil lifting  $\widehat{\Delta} = \widehat{P}_{\lambda_0}^\rho(\Delta_{\lambda_0})$  by changing partial derivatives  $\partial_i$  for the covariant derivatives  $\nabla_i = \partial_i + (\widehat{\lambda} - \lambda_0) \Gamma_i$ . The operator  $\widehat{\Delta}$  is an operator on the algebra of densities of the same order as the operator  $\Delta_{\lambda_0}$ .

**Remark 5.** The above expression still makes sense when  $\Gamma_i$  is an arbitrary connection. We shall only consider the above case when  $\Gamma_i$  corresponds to a volume form. (Note that in the case  $\Gamma_i$  is flat the map (20) is not just a linear map but a map of algebras.)

It is important to note that the adjointness operation (9) commutes with the identification isomorphisms  $P_{\lambda_1, \lambda_2}^\rho$ :

$$\forall \Delta \in \mathcal{D}_{\lambda_0}(M) \quad (P_{\lambda, \lambda_0}^\rho(\Delta))^* = P_{1-\lambda, 1-\lambda_0}^\rho(\Delta^*),$$

and for the canonical lifting (19)

$$\forall \Delta \in \mathcal{D}_{\lambda_0}(M) \quad \left( \widehat{P}_{\lambda_0}^\rho(\Delta) \right)^* = \widehat{P}_{1-\lambda_0}^\rho(\Delta^*). \quad (21)$$

(Recall that  $\Delta^* \in \mathcal{D}_{1-\lambda}(M)$  if  $\Delta \in \mathcal{D}_\lambda(M)$ .)

On manifolds with a volume form structure using the identification isomorphisms  $P_{\lambda_1, \lambda_2}^\rho$  one can consider not only the canonical scalar product (8) on  $\mathcal{F}(M)$ , but also the scalar product defined on densities of fixed weights: for  $\mathbf{s}_1, \mathbf{s}_2 \in \mathcal{F}_\lambda(M)$ ,  $\langle \mathbf{s}_1, \mathbf{s}_2 \rangle_\rho = \int \mathbf{s}_1 \cdot \mathbf{s}_2 \cdot \rho^{1-2\lambda}$ . Similarly for an operator  $\Delta \in \mathcal{D}_\lambda(M)$  one can consider its adjoint (with respect to the volume form  $\rho$ ), such that it acts on densities of weight  $\lambda$  also:  $\Delta^{*\rho} = P_{\lambda, 1-\lambda}^\rho(\Delta^*)$ . In particular if an operator  $\Delta$  acts on usual functions then its canonical adjoint (9)  $\Delta^*$  acts on densities of weight 1 whilst its adjoint with respect to the volume form  $\rho$  is an operator acting on functions:  $\Delta^{*\rho} = P_{0,1}^\rho(\Delta^*) = \rho^{-1} \circ \Delta^* \circ \rho$ .

We shall now state the following lemma whose proof we reserve for the appendix.

**Lemma 1.** *Let  $M$  be a manifold provided with a volume form  $\rho$ . Denote by  $\mathcal{D}(M)$  the space of linear differential operators on functions on  $M$  and by  $\mathcal{D}^{(n)}(M)$  its subspace of differential operators of order  $\leq n$ . Let  $F$  be the linear map defined by*

$$F(\Delta) = a\Delta + b\Delta^{*\rho} + c\Delta(1) + d\Delta^{*\rho}(1) \quad (22)$$

*mapping  $\mathcal{D}(M)$  to itself, where  $a, b, c, d$  are constants, and  $\Delta^{*\rho}$  is the operator adjoint to  $\Delta$  with respect to volume form  $\rho$ . Then this map is  $\text{SDiff}_\rho(M)$ -equivariant, and moreover the converse implication is true in the case that  $M$  is connected manifold of dimension  $\geq 3$ : if  $F$  is an arbitrary linear  $\text{sdiff}_\rho(M)$ -equivariant map defined on the subspace  $\mathcal{D}^{(n)}(M)$  with values in the space  $\mathcal{D}(M)$ , then  $F(\Delta)$  has the appearance (22). (We suppose that  $F(\Delta)$  is a differential polynomial on the coefficients of the operator  $\Delta$ .)*

**Remark 6.** If our map  $F$  obeys the stronger condition that it is equivariant with respect to the whole algebra  $\text{diff}(M)$  of infinitesimal diffeomorphisms then we have that

$$F(\Delta) = a\Delta + c\Delta(1). \quad (23)$$

The statement of the lemma and equation (23) are a sort of Schur lemma for the group of diffeomorphisms and its subgroup  $\text{SDiff}_\rho(M)$ . Equation (23) states that the  $\text{Diff}(M)$ -equivariant linear maps on operators on functions is proportional to the identity operator on the two invariant subspaces of normalised operators and operators of multiplication by functions. Equation (22) of the lemma states that a  $\text{SDiff}_\rho(M)$ -equivariant linear map on  $\mathcal{D}(M)$  is proportional to the identity operator on the invariant spaces of normalised self-adjoint operators and normalised anti-self-adjoint operators. (Operator  $\Delta$  is normalised if  $\Delta(1) = 0$ .)

We shall use the lemma for constructing pencil liftings which are  $\text{sdiff}(M)$  and  $\text{diff}(M)$ -equivariant. Firstly we consider  $\text{sdiff}(M)$ -equivariant pencil liftings.

Let a manifold  $M$  be equipped with a volume form  $\rho$ . The pencil lifting (19) assigns to any operator  $\Delta \in \mathcal{D}_{\lambda_0}^{(n)}(M)$  the operator  $\widehat{\Delta} = \widehat{P}_{\lambda_0}^\rho(\Delta)$  of the same order on algebra of densities  $\mathcal{F}(M)$ , i.e. the operator pencil  $\{\Delta_\lambda\}: \Delta_\lambda = P_{\lambda, \lambda_0}^\rho(\Delta)$ . This pencil passes through the operator  $\Delta$ ,  $\widehat{\Delta}|_{\widehat{\lambda}=\lambda_0} = \Delta$ . Thus the canonical lifting (19) is  $\text{sdiff}_\rho(M)$ -lifting which is strictly regular. Are there another strictly regular or just regular  $\text{sdiff}_\rho(M)$ -liftings? To answer this question we analyze regular  $\text{sdiff}_\rho(M)$ -lifting maps using lemma 1.

Pick arbitrary  $n$  and  $\lambda_0$  and let  $\widehat{\Pi} = \widehat{\Pi}_{\lambda_0}^{\rho, (n)}(\Delta)$  be a regular  $\text{sdiff}_\rho(M)$ -lifting map defined on the space  $\mathcal{D}_{\lambda_0}^{(n)}(M)$ . Recall that a regular lifting on  $\mathcal{D}_{\lambda_0}^{(n)}(M)$  is a linear map on  $\mathcal{D}_{\lambda_0}^{(n)}(M)$  which takes values in the space  $\widehat{\mathcal{D}}^{(n)}(M)$  of operators of order  $\leq n$  on  $\mathcal{F}(M)$  and a strictly

regular lifting of  $\mathcal{D}_{\lambda_0}^{(n)}(M)$  restricted to the subspaces  $\mathcal{D}_{\lambda_0}^{(k)}(M)$  takes values in the subspaces  $\widehat{\mathcal{D}}^{(k)}(M)$  for all  $k \leq n$ .

Using the identification isomorphisms (18) assign to the lifting  $\widehat{\Pi}_{\lambda_0}^{\rho, (n)}$  the pencil  $\{F_{\lambda}^{\widehat{\Pi}}\}$  of differential operators acting on functions, defined by  $F_{\lambda}^{\widehat{\Pi}} = P_{0, \lambda}^{\rho} \circ \left( \widehat{\Pi}_{\lambda_0}^{\rho, (n)} \right) |_{\widehat{\lambda}=\lambda} \circ P_{\lambda_0, 0}^{\rho}$ , i.e. for an arbitrary operator  $\Delta$  on functions

$$F_{\lambda}^{\widehat{\Pi}}(\Delta) = \rho^{-\lambda} \circ \widehat{\Pi}_{\lambda_0}^{\rho, (n)} |_{\widehat{\lambda}=\lambda} (\rho^{\lambda_0} \circ \Delta \circ \rho^{-\lambda_0}) \circ \rho^{\lambda}.$$

All the maps  $F_{\lambda}^{\widehat{\Pi}}$  are linear,  $\text{sdiff}_{\rho}(M)$ -equivariant maps since the lifting  $\widehat{\Pi}$  is a  $\text{sdiff}_{\rho}(M)$ -lifting. Using equation (22) of lemma 1 applied to the pencil  $\{F_{\lambda}^{\widehat{\Pi}}\}$ , the relations (21), and the fact that the map  $\widehat{\Pi} = \widehat{\Pi}_{\lambda_0}^{\rho, (n)}$  is regular, we see that in the case if dimension of  $M$  is greater than 2 then

$$\widehat{\Pi}(\Delta) = A(\widehat{\lambda}) \widehat{P}_{\lambda_0}^{\rho}(\Delta) + B(\widehat{\lambda}) \left( \widehat{P}_{\lambda_0}^{\rho}(\Delta) \right)^* + C(\widehat{\lambda}) \widehat{P}_{\lambda_0}^{\rho}(1) + D(\widehat{\lambda}) \left( \widehat{P}_{\lambda_0}^{\rho}(\Delta) \right)^*(1). \quad (24)$$

Here  $A(\widehat{\lambda}), B(\widehat{\lambda}), C(\widehat{\lambda})$  and  $D(\widehat{\lambda})$  are vertical operators (6), that are polynomials in  $\widehat{\lambda}$  of the form:

$$A(\widehat{\lambda}) = 1 - b(\widehat{\lambda} - \lambda_0), \quad B(\widehat{\lambda}) = (-1)^n b(\widehat{\lambda} - \lambda_0), \quad C(\widehat{\lambda}) = \sum_{k=1}^n c_k (\widehat{\lambda} - \lambda_0)^k, \quad D(\widehat{\lambda}) = \sum_{k=1}^n d_k (\widehat{\lambda} - \lambda_0)^k,$$

where  $\{b, c_k, d_k\}$ ,  $k = 1, 2, \dots, n$  are constants. Note that difference of two arbitrary lifting maps (24) belong to linear space of liftings which vanish the space  $\mathcal{D}_{\lambda_0}^{(n)}$ . We come to the following proposition

**Proposition 1.** 1. All linear maps of the form (24) are regular  $\text{SDiff}_{\rho}(M)$ -liftings defined on the space  $\mathcal{D}_{\lambda_0}^{(n)}(M)$ . If  $n \geq 2$ , then the space of these liftings is an affine space of dimension  $2n + 1$ . Its dimension is equal to 2 if  $n = 1$  (see the example 3.1 below for more detail).

2. If  $M$  is a connected manifold of dimension  $\geq 3$  then an arbitrary regular  $\text{sdiff}_{\rho}(M)$ -equivariant lifting of  $\mathcal{D}_{\lambda_0}^{(n)}$  belongs to this affine space

3. If in equation (24)  $b = 0$ ,  $d_1 = -c_1$  and  $d_i = c_i = 0$  for all  $i \geq 2$  then we come to an affine line of strictly regular  $\text{SDiff}_{\rho}(M)$ -liftings. If  $M$  is a connected manifold of dimension  $\geq 3$  then an arbitrary strictly regular  $\text{sdiff}_{\rho}(M)$ -equivariant liftings of  $\mathcal{D}_{\lambda_0}^{(n)}$  belongs to this affine line.

**Remark 7.** The affine space (24) of  $\text{sdiff}_{\rho}(M)$ -regular lifting maps has the following natural flag structure:

- point —  $A = 1, B = C = D = 0$ , i.e. the strictly regular lifting  $\Pi_{\lambda_0}^{\rho, (n)}(\Delta) = P_{\lambda_0}^{\rho}(\Delta)$ ,
- line —  $C = D = 0$ , i.e. the 1-parameter family  $\widehat{\Pi}_b$  of regular liftings

$$\widehat{\Pi}_b(\Delta) = (1 - b(\widehat{\lambda} - \lambda_0)) \widehat{P}_{\lambda_0}^{\rho}(\Delta) + (-1)^n b(\widehat{\lambda} - \lambda_0) \left( \widehat{P}_{\lambda_0}^{\rho}(\Delta) \right)^*, \quad b \in \mathbb{R}. \quad (25)$$

In the previous section we considered canonical liftings for first order operators, in the following example we will use the proposition to describe all regular  $\text{sdiff}(M)$ -equivariant lifings for first order operators and distinguish those that are  $\text{diff}(M)$ -equivariant. As a consequence we will describe all  $\text{diff}(M)$ -liftings for first order operators since any  $\text{diff}(M)$ -lifting is necessarily a  $\text{sdiff}(M)$ -lifting.

**Example 3.1.** Choose an arbitrary volume form  $\rho$  on  $M$  and pick an arbitrary  $\lambda_0$ . Let  $\Delta$  be a first order operator acting on densities of weight  $\lambda_0$ ,  $\Delta \in \mathcal{D}_{\lambda_0}^{(1)}(M)$ . According to equation (20) for the canonical liftings  $\widehat{P}_{\lambda_0}^\rho$  (19) we have that

$$\widehat{\Delta} = \widehat{P}_{\lambda_0}^\rho(\Delta) = A^i(x) \left( \partial_i + (\widehat{\lambda} - \lambda_0)\Gamma_i(x) \right) + B(x), \quad (\Delta = A^i(x)\partial_i + B(x)),$$

where as usual  $\Gamma_i(x) = -\partial_i \log \rho(x)$  ( $\rho = \rho(x)|Dx|$ ) are components of a flat connection in the local coordinates  $x^i$ . Hence using equation (24) we come to the following family of regular sdiff  $\rho(M)$ -liftings

$$\begin{aligned} \widehat{\Pi}(\Delta) &= (1-b(\widehat{\lambda}-\lambda_0))\widehat{P}_{\lambda_0}^\rho(\Delta) - b(\widehat{\lambda}-\lambda_0) \left( \widehat{P}_{\lambda_0}^\rho(\Delta) \right)^* + c(\widehat{\lambda}-\lambda_0)\widehat{P}_{\lambda_0}^\rho(1) + d(\widehat{\lambda}-\lambda_0) \left( \widehat{P}_{\lambda_0}^\rho(\Delta) \right)^* (1) = \\ &= A^i(x)\partial_i + B(x) + (\widehat{\lambda} - \lambda_0) (k_1 B(x) + k_2 \partial_i A^i(x) + (1 - \lambda_0 k_1 - k_2) A^i(x)\Gamma_i(x)), \quad k_1, k_2 \in \mathbb{R} \end{aligned} \quad (26)$$

where we denote by  $k_1 = c + d - 2b$  and  $k_2 = b - d$ . We come to a 2-parametric family of regular sdiff  $\rho(M)$ -liftings, i.e an affine plane of liftings. It is evident that the above lifting is strictly regular lifting if  $k_1 = 0$ .

Now we can calculate all regular diff  $(M)$ -lifting maps on  $\mathcal{D}_{\lambda_0}^{(1)}(M)$ . We have that a regular diff  $(M)$ -equivariant lifting is necessarily a sdiff  $(M)$ -lifting that does not depend on the volume form. One can see from equation (26) that the regular sdiff  $(M)$ -liftings that do not depend on the volume form if  $k_2 + \lambda_0 k_1 - 1 = 0$ , i.e.  $k_2 = 1 - \lambda_0 k_1$ . Thus we come to an affine line of diff  $(M)$ -regular liftings:

$$\widehat{\Pi}(\Delta) = A^i \partial_i + \widehat{\lambda} \partial_i A^i(x) + \left( 1 + k_1 (\widehat{\lambda} - \lambda_0) \right) (B(x) - \lambda_0 \partial_i A^i), \quad k_1 \in \mathbb{R},$$

which is just the family of canonical liftings (13)). In the case that  $k_1 = 0$  we come to the strictly regular diff  $(M)$ -lifting (compare with the canonical liftings (12)), and finally in the case if  $k_1 = 2/(2\lambda_0 - 1)$  we come to the anti-self-adjoint lifting (14) (for  $\lambda_0 \neq \frac{1}{2}$ ).

We see that that diff  $(M)$ -equivariant liftings of first order operators are exhausted by the canonical liftings considered in the section 2.1. Later we shall apply these methods to the analysis of liftings for higher order operators.

**Example 3.2.** Consider liftings of second order operators acting on densities of weight  $\lambda_0$  and let  $\Delta = S^{ij}(x)\partial_i\partial_j + T^i(x)\partial_i + R(x) \in \mathcal{D}_{\lambda_0}^{(2)}(M)$ . Its adjoint is of the form

$$\Delta^* = S^{ij}(x)(x)\partial_i\partial_j + (2\partial_r S^{ri}(x) - T^i)(x)\partial_i + (R(x) - \partial_r T^r(x) + \partial_r \partial_q S^{rq}(x))$$

acting on densities of weight  $1 - \lambda_0$  (see (9)). Consider the lifting (19) of both these operators:

$$\begin{aligned} \widehat{P}_{\lambda_0}^\rho(\Delta) &= \rho^{\widehat{\lambda}-\lambda_0} \circ \Delta \circ \rho^{\lambda_0-\widehat{\lambda}} = S^{ij}(x)\partial_i\partial_j + 2(\widehat{\lambda} - \lambda_0)\Gamma^i(x)\partial_i + T^i(x)\partial_i + \\ &+ (\widehat{\lambda} - \lambda_0)^2 \Gamma^i(x)\Gamma_i(x) + (\widehat{\lambda} - \lambda_0)\partial_i \Gamma^i(x) + (\widehat{\lambda} - \lambda_0)(T^i(x) - \partial_r S^{ri}(x))\Gamma_i(x) + R(x), \\ \left( \widehat{P}_{\lambda_0}^\rho(\Delta) \right)^* &= \left( \widehat{P}_{1-\lambda_0}^\rho(\Delta^*) \right) = S^{ij}(x)\partial_i\partial_j + 2(\widehat{\lambda} + \lambda_0 - 1)\Gamma^i(x)\partial_i + (2\partial_r S^{ri}(x) - T^i(x))\partial_i + \\ &+ (\widehat{\lambda} + \lambda_0 - 1)^2 \Gamma^i(x)\Gamma_i(x) + (\widehat{\lambda} + \lambda_0 - 1)\partial_i \Gamma^i(x) + (1 - \widehat{\lambda} - \lambda_0)T^i(x)\Gamma_i(x) + R(x) - \partial_r T^r(x) + \partial_r \partial_q S^{rq}(x). \end{aligned}$$

Here  $\Gamma_i(x) = -\partial_i \log \rho(x)$  and  $\Gamma^i(x) = S^{ik}(x)\Gamma_k(x)$ .

For the rest of this example, to ease the calculations, we will suppose that local coordinate system  $(x^i)$  are normal coordinates with respect to the volume form  $\rho$ , i.e.  $\rho$  is the

coordinate volume form in these coordinates  $\rho = |Dx|$ . In particular in normal coordinates the connection  $\Gamma_i(x) = -\partial_i \log \rho(x)$  vanishes. In this case the above formulae become far simpler and we see that the regular SDiff  $(M)$ -lifting map (24) has the appearance

$$\widehat{\Pi}(\Delta) = \Delta + 2b(\widehat{\lambda} - \lambda_0) (\partial_r S^{ri}(x) - T^i(x)) \partial_i + b(\widehat{\lambda} - \lambda_0) (\partial_r \partial_i S^{ri}(x) - \partial_i T^i(x)) \partial_i + (\widehat{\lambda} - \lambda_0) \left[ (c_1 + c_2(\widehat{\lambda} - \lambda_0)) R(x) + (d_1 + d_2(\widehat{\lambda} - \lambda_0)) (\partial_r \partial_q S^{rq}(x) - \partial_i T^i(x) + R(x)) \right].$$

This regular lifting depends on 5 parameters and in general it is not strictly regular. For example take the first order operator  $\Delta = T^i(x) \partial_i + R(x) \in \mathcal{D}_{\lambda_0}^{(2)}(M)$  ( $S^{ij}(x) \equiv 0$ ), then the lift is of the form  $\widehat{\Pi}(\Delta) = \Delta - 2b(\widehat{\lambda} - \lambda_0) T^i(x) \partial_i + \dots$ , which is an operator of order 2 (if  $b \neq 0$  and  $T^i(x) \neq 0$ ). We find that this lifting is strictly regular if  $b = c_2 = d_2 = 0$  and  $c_1 = -d_1$ :  $\Pi(\Delta) = \Delta + d(\widehat{\lambda} - \lambda_0) (\partial_i T^i - \partial_r \partial_q S^{rq})$ .

#### 4. SELF-ADJOINT $\text{sdiff}_{\rho}(M)$ -LIFTINGS AND $\text{diff}(M)$ -LIFTINGS

To use the results obtained above it is useful to think of  $\text{diff}(M)$ -liftings as  $\text{sdiff}_{\rho}(M)$ -liftings which are independent of the volume form  $\rho$ . We will use this fact to describe  $\text{diff}(M)$ -liftings for higher order operators in this section. (For first order operators we did this in example 3.1.)

First we consider  $\text{sdiff}(M)$ -liftings, which are self-adjoint or anti-self-adjoint, depending on whether the number  $n$  is even or odd (see remark 2). The condition of (anti)-self-adjointness arises naturally when one tries to minimise the dependance of lifting maps on a volume form.

We show that if a lifting is invariant under a variation of the volume form (up to a vertical map) then it is (anti)-self-adjoint. This will lead us to give an alternative proof of existence and uniqueness of  $\text{diff}(M)$ -lifting for second order operators (this is just the self-adjoint lifting described in subsection 2.2) and we will show that there are no  $\text{diff}(M)$ -liftings of operators of order  $\geq 3$  for manifolds of dimension  $\geq 3$ .

**4.1. Self-adjoint and anti-self-adjoint  $\text{sdiff}(M)$ -liftings.** As usual choose an arbitrary volume form  $\rho$  on manifold  $M$ . Pick an arbitrary  $n$  and  $\lambda_0$ . In the flag of regular  $\text{sdiff}_{\rho}$ -liftings of space  $\mathcal{D}_{\lambda_0}^{(n)}(M)$  consider the line of liftings (25),  $\widehat{\Pi}_b = \widehat{\Pi}_{\lambda_0, b}^{\rho, (n)}(\Delta)$ ,  $b \in \mathbb{R}$  (see remark 7). In the case that  $n$  is an even number, choose, in this family, the self-adjoint lifting of the space  $\mathcal{D}_{\lambda_0}^{(n)}$  (in the case where  $n$  is odd we choose the anti-self-adjoint lifting). The condition that  $(\widehat{\Pi}_b(\Delta))^* = (-1)^n \widehat{\Pi}_b(\Delta)$  implies that  $(1 - b(\widehat{\lambda} - \lambda_0))^* = b(\widehat{\lambda} - \lambda_0)$  i.e.

$$b = \frac{1}{1 - 2\lambda_0}. \quad (27)$$

Hence in the case that  $\lambda_0 \neq \frac{1}{2}$ , we come to a distinguished pencil lifting map of  $\mathcal{D}_{\lambda_0}^{(n)}$  belonging to the affine line (25):

$$\widehat{\Pi}_{\text{disting.}}(\Delta) = \frac{\widehat{\lambda} + \lambda_0 - 1}{2\lambda_0 - 1} \widehat{P}_{\lambda_0}^{\rho}(\Delta) + (-1)^n \frac{\lambda_0 - \widehat{\lambda}}{2\lambda_0 - 1} \left( \widehat{P}_{\lambda_0}^{\rho}(\Delta) \right)^*, \quad (28)$$

which is (anti)-self-adjoint:  $(\widehat{\Pi}_{\text{disting.}}(\Delta))^* = (-1)^n \widehat{\Pi}_{\text{disting.}}(\Delta)$ .

**Example 4.1.** Take a 3-rd order operator

$$\Delta = S^{ikm} \partial_i \partial_k \partial_m + G^{ik} \partial_i \partial_k + A^i \partial_i + R,$$

acting on densities of weight  $\lambda_0$ ,  $\Delta \in \mathcal{D}_{\lambda_0}^{(n)}$  where  $\lambda_0 \neq \frac{1}{2}$ . We choose normal coordinates, i.e. coordinates such that volume form  $\rho = |Dx|$ . Then

$$\begin{aligned} \Pi_{\text{disting.}}(\Delta) &= \frac{\widehat{\lambda} + \lambda_0 - 1}{2\lambda_0 - 1} \widehat{P}_{\lambda_0}^{\rho}(\Delta) - \frac{\lambda_0 - \widehat{\lambda}}{2\lambda_0 - 1} \left( \widehat{P}_{\lambda_0}^{\rho}(\Delta) \right)^* = \\ &= \frac{\widehat{\lambda} + \lambda_0 - 1}{2\lambda_0 - 1} (S^{ikm} \partial_i \partial_k \partial_m + G^{ik} \partial_i \partial_k + \dots) + \frac{\widehat{\lambda} - \lambda_0}{2\lambda_0 - 1} (-S^{ikm} \partial_i \partial_k \partial_m + (G^{km} - 3\partial_i S^{ikm}) \partial_k \partial_m + \dots) = \\ &= S^{ikm} \partial_i \partial_k \partial_m + 3 \frac{\widehat{\lambda} - \lambda_0}{2\lambda_0 - 1} \partial_i S^{ikm} \partial_k \partial_m + \frac{2\widehat{\lambda} - 1}{2\lambda_0 - 1} G^{km} \partial_k \partial_m + \dots \end{aligned}$$

(we denote by dots operators of order  $\leq 1$ ).

Now consider an arbitrary (anti)-self-adjoint liftings in the affine space (24), which differ from the distinguished lifting (28) by a vertical map  $C(\widehat{\lambda}) \widehat{P}_{\lambda_0}^{\rho}(\Delta)(1) + D(\widehat{\lambda}) \left( \widehat{P}_{\lambda_0}^{\rho}(\Delta) \right)^*$  (1) (see equation (24)). The polynomials  $C(\widehat{\lambda})$  and  $D(\widehat{\lambda})$  in equation (24) have to be self-adjoint vertical operators if  $n$  is even, respectively anti-self-adjoint if  $n$  is odd. One can see that these polynomials have the following appearance:

$$C(\widehat{\lambda}) = t^{p(n)} \sum_{k=1}^{\frac{n-p(n)}{2}} c_k \left( t^{2k}(\widehat{\lambda}) - t^{2k}(\lambda_0) \right), \quad D(\widehat{\lambda}) = t^{p(n)} \sum_{k=1}^{\frac{n-p(n)}{2}} d_k \left( t^{2k}(\widehat{\lambda}) - t^{2k}(\lambda_0) \right), \quad (29)$$

where  $t(\widehat{\lambda}) = \widehat{\lambda} - \frac{1}{2}$  is anti-self-adjoint linear polynomial in  $\widehat{\lambda}$ :  $t^*(\widehat{\lambda}) = \left( \widehat{\lambda} - \frac{1}{2} \right)^* = -\widehat{\lambda} + \frac{1}{2} = -t(\widehat{\lambda})$ ,  $p(n) = 0$  if  $n$  is even and  $p(n) = 1$  if  $n$  is odd.

**Proposition 2.** For  $\lambda_0 \neq \frac{1}{2}$  and  $n \geq 2$  the  $2n + 1$ -dimensional affine space of regular sdiff  $\rho(M)$ -liftings of the space  $\mathcal{D}_{\lambda_0}^{(n)}$  (24) possesses a subplane of self-adjoint (anti-self-adjoint) liftings of dimension  $n$  (of dimension  $n-1$ ) if  $n$  is even (if  $n$  is odd) described by the equations (29).

The affine line (25) of sdiff  $(M)$ -liftings possesses a distinguished lifting  $\widehat{\Pi}_{\text{disting.}}(\Delta)$  (28). This lifting is self-adjoint if  $n$  is even, and it is anti-self-adjoint if  $n$  is odd.

**Example 4.2.** In example 3.2 we considered regular liftings for second order operators. We now consider self-adjoint regular liftings of these operators. For an arbitrary second order operator acting on densities of weight  $\lambda_0$ ,  $\Delta = S^{ij}(x) \partial_i \partial_j + T^i(x) \partial_i + R(x) \in \mathcal{D}_{\lambda_0}^{(2)}$  ( $\lambda_0 \neq 1/2$ ) according to the proposition self-adjoint lifting  $\widehat{\Pi}$  has the following appearance:

$$\widehat{\Pi}(\Delta) = \widehat{\Pi}_{\text{disting.}}(\Delta) + \left( t^2(\widehat{\lambda}) - t^2(\lambda_0) \right) (cR + d(\partial_r \partial_i S^{ri} - \partial_i T^i + R)).$$

Using the equations for regular liftings from example 3.2 and equation (28), we have that

$$\widehat{\Pi}_{\text{disting.}}(\Delta) = \frac{\widehat{\lambda} + \lambda_0 - 1}{2\lambda_0 - 1} \widehat{P}_{\lambda_0}^{\rho}(\Delta) + \frac{\lambda_0 - \widehat{\lambda}}{2\lambda_0 - 1} \left( \widehat{P}_{\lambda_0}^{\rho}(\Delta) \right)^*$$

and

$$\begin{aligned} \widehat{\Pi}(\Delta) = & S^{ij}(x)\partial_i\partial_j + \partial_r S^{ri}\partial_i + \frac{2\widehat{\lambda} - 1}{2\lambda_0 - 1}(T^i - \partial_r S^{ri}) + \frac{\widehat{\lambda} - \lambda_0}{2\lambda_0 - 1}(\partial_i T^i - \partial_r \partial_i S^{ri}) + R + \\ & + \left( \widehat{\lambda}(\widehat{\lambda} - 1) - \lambda_0(\lambda_0 - 1) \right) (cR + d(\partial_r \partial_i S^{ri} - \partial_i T^i + R)) , \end{aligned} \quad (30)$$

where  $c, d$  are arbitrary constants (We work in normal coordinates such that volume form  $\rho = |Dx|$ ). We come to the plane (2-dimensional space) of self-adjoint regular sdiff  $\rho$  lifting maps of  $\mathcal{D}_{\lambda_0}^{(2)}(M)$  for  $\lambda_0 \neq 1/2$ . Note that in the case if  $\lambda_0 \neq 0, 1$  this lifting map can be presented as the sum of canonical map (15) ( $F = 0$ ) and vertical map

$$\widehat{\Pi}(\Delta) = \left( \widehat{\lambda}(\widehat{\lambda} - 1) - \lambda_0(\lambda_0 - 1) \right) ((c + d)\lambda_0(\lambda_0 - 1) - 1) \theta (c - d)\lambda_0 + d) \partial_i \gamma^i ,$$

where  $\gamma^i, \theta$  are upper connection and Brans-Dicke scalar corresponding to operator  $\Delta$  (see section 2.2).

**4.2. Dependence of sdiff  $(M)$ -liftings and (anti)-self-adjoint liftings on the volume form.** Now we shall calculate the infinitesimal variation of regular sdiff  $(M)$ -equivariant liftings (24) with respect to an infinitesimal variation of the volume form.

Pick an arbitrary  $n$  and  $\lambda_0$  and choose within the affine space of liftings (24) an arbitrary regular sdiff  $\rho(M)$ -lifting,  $\widehat{\Pi}$ . We shall calculate the variation of this lifting with respect to the variation of the volume form,  $\rho \rightarrow \rho + \delta\rho$ , firstly by considering the canonical sdiff  $\rho(M)$ -lifting  $\widehat{P}_{\lambda_0}^\rho$ , (see (19)) which has the following variation:

$$\delta_\rho \widehat{P}_{\lambda_0}^\rho = \left( \widehat{\lambda} - \lambda_0 \right) \left( \rho^{-1} \delta\rho \circ \widehat{P}_{\lambda_0}^\rho - \widehat{P}_{\lambda_0}^\rho \circ \delta\rho^{-1} \rho \right) = \left( \widehat{\lambda} - \lambda_0 \right) \left[ \rho^{-1} \delta\rho, \widehat{P}_{\lambda_0}^\rho \right] . \quad (31)$$

(If  $\rho = \rho_t$  is a one-parametric family of volume forms, then  $\dot{\rho}_t = \rho_t(\rho_t^{-1} \dot{\rho}_t)$ , where  $\rho_t^{-1} \dot{\rho}_t$  is a scalar function and  $\delta\rho = \dot{\rho}_t|_{t=0} \delta t$ .)

Any lifting (24) is the sum of a lifting  $\widehat{\Pi}_b$  belonging to the affine line (25), and a vertical map (see Proposition 1). The variation of a vertical map with respect to volume form is a vertical map, hence we have that the variation of the regular sdiff  $\rho$ -lifting (24) with respect to volume form  $\rho$  is equal to

$$\begin{aligned} \delta_\rho \Pi: \quad \delta_\rho \Pi(\Delta) = & \delta_\rho \left( \widehat{\Pi}_b(\Delta) \right) + \text{variation of vertical map} = \\ & A(\widehat{\lambda}) \left( \widehat{\lambda} - \lambda_0 \right) \left[ \rho^{-1} \delta\rho, \widehat{P}_{\lambda_0}^\rho(\Delta) \right] + B(\widehat{\lambda}) \left( \widehat{\lambda} + \lambda_0 - 1 \right) \left[ \rho^{-1} \delta\rho, \widehat{P}_{\lambda_0}^{\rho*}(\Delta) \right] + \text{vertical map} \\ & = (1 + b(2\lambda_0 - 1)) \left( \widehat{\lambda} - \lambda_0 \right) \left[ \rho^{-1} \delta\rho, \widehat{P}_{\lambda_0}^\rho(\Delta) \right] - \\ & + b(1 - \widehat{\lambda} - \lambda_0) \underbrace{\left( \widehat{\lambda} - \lambda_0 \right) \left[ \rho^{-1} \delta\rho, P_{\lambda_0}^\rho(\Delta) - (-1)^n (P_{\lambda_0}^\rho)^*(\Delta) \right]}_{\text{operator of order } \leq n - 2} + \text{vertical map} \end{aligned} \quad (32)$$

(Recall that  $(\widehat{\lambda} - \lambda_0)^* = (1 - \widehat{\lambda} - \lambda_0)$ , the operator  $P_{\lambda_0}^\rho(\Delta) - (-1)^n (P_{\lambda_0}^\rho(\Delta))^*$  has order  $\leq n - 1$ , and the commutator of two operators of orders  $m, n$  has order  $m + n - 1$ .)

It follows from this relation that the condition  $b = \frac{1}{1-2\lambda_0}$  (if  $\lambda_0 \neq 1/2$ ) (27), which means that the lifting is the distinguished lifting up to a vertical map, is a necessary condition for the lifting to be independent of infinitesimal variations of the volume form (up to a vertical maps). We come to the following proposition:



**Proposition 3.** *Choose a volume form  $\rho$  on manifold  $M$ .*

1. *Let  $\Pi = \Pi_{\text{disting.}}$  be the distinguished sdiff  $\rho(M)$ -lifting (28) defined on the space  $\mathcal{D}_{\lambda_0}^{(n)}(M)$  ( $\lambda_0 \neq 1/2$ ).*

*Then*

$$\delta_\rho \Pi(\Delta) = \frac{(\widehat{\lambda} - \lambda_0)(\widehat{\lambda} + \lambda_0 - 1)}{2\lambda_0 - 1} \left[ \rho^{-1} \delta \rho, \widehat{P}_{\lambda_0}^\rho(\Delta) - (-1)^n \left( \widehat{P}_{\lambda_0}^\rho(\Delta)^* \right) \right]. \quad (33)$$

2. *Let  $\widehat{\Pi}$  be an arbitrary regular sdiff  $\rho(M)$ -equivariant lifting defined on the space  $\mathcal{D}_{\lambda_0}^{(n)}(M)$ . Suppose that  $n \geq 2$ , and let  $\Delta_0$  in  $\mathcal{D}_{\lambda_0}^{(n)}(M)$  be an operator of order  $n$ , i.e.  $\Delta_0 \notin \mathcal{D}_{\lambda_0}^{(n-1)}(M)$ , such that the infinitesimal variation with respect to the volume form  $\rho$  of  $\Pi$  at  $\Delta_0$  vanishes:*

$$\delta_\rho \Pi|_{\Delta=\Delta_0} = 0.$$

*Then in the case the dimension of  $M$  is greater than 2*

- $\lambda_0 \neq \frac{1}{2}$  since in the case of  $\lambda_0 = \frac{1}{2}$  the sdiff -lifting of  $n$ -th order operator essentially depends on the volume form.
- the lifting  $\Pi$  is equal to the distinguished lifting (28) up to vertical maps:

$$\widehat{\Pi}(\Delta) = \widehat{\Pi}_{\text{disting.}}(\Delta) + \text{vertical map}, \quad \widehat{\Pi}^*(\Delta) = (-1)^n \widehat{\Pi}(\Delta) + \text{vertical map}.$$

*In particular the canonical lifting of the operator  $\Delta_0$  is (anti)-self-adjoint up to a vertical operator.*

Namely if an operator  $\Delta_0$  has order  $n$ , ( $\Delta_0 \in \mathcal{D}_{\lambda_0}^{(n)}(M)$ , and  $\Delta_0 \notin \mathcal{D}_{\lambda_0}^{(n-1)}(M)$ ) then due to equation (32) the infinitesimal variation of the canonical lifting  $P_{\lambda_0}^\rho$  at  $\Delta_0$  does not vanish,  $[\delta_\rho, P_{\lambda_0}^\rho(\Delta_0)] \neq 0$ , if  $(1 + b(2\lambda_0 - 1)) \neq 0$  i.e. if condition (27) is not obeyed, and the pencil is not the distinguished pencil. In particular if  $\lambda_0 = 1/2$  then the variation does not vanish for any choice of  $b$ . If  $\lambda_0 \neq 1/2$  then the variation has a chance to vanish only for the distinguished pencil.

This proposition is crucial for extracting diff  $(M)$ -equivariant liftings from the class of sdiff  $(M)$ -liftings. To this end let  $\Pi^\rho$  be an arbitrary sdiff  $\rho$ -lifting (24) and let  $\mathbf{X}$  be an arbitrary vector field on  $M$ . We can express the Lie derivative  $\widehat{\text{ad}}_{\mathbf{X}}$  of the lifting  $\Pi^\rho$  in terms of the variation with respect to the volume form. The Lie derivative,  $\text{ad}_{\mathbf{X}}$ , of an arbitrary operator is equal to  $\text{ad}_{\mathbf{X}}\Delta$ :  $\text{ad}_{\mathbf{X}}\Delta(\mathbf{s}) = \mathcal{L}_{\mathbf{X}}(\Delta(\mathbf{s})) - (\Delta(\mathcal{L}_{\mathbf{X}}\mathbf{s}))$  and for maps on operators

$$\widehat{\text{ad}}_{\mathbf{X}}\widehat{\Pi}: \widehat{\text{ad}}_{\mathbf{X}}\widehat{\Pi}(\Delta) = \text{ad}_{\mathbf{X}}\left(\widehat{\Pi}(\Delta)\right) - \left(\widehat{\Pi}(\text{ad}_{\mathbf{X}}\Delta)\right).$$

If  $\widehat{\Pi} = \widehat{\Pi}^\rho$  is a map of operators depending on the volume form  $\rho$ , such that for an arbitrary volume form it is sdiff  $\rho(M)$ -equivariant, then for a vector field  $\mathbf{X}$  we arrive at the Lie derivative from the variation by setting  $\delta\rho$  equal to  $\mathcal{L}_{\mathbf{X}}\rho = \rho \text{div}_\rho \mathbf{X}$ :

$$\widehat{\text{ad}}_{\mathbf{X}}\widehat{\Pi}^\rho = \delta_\rho \widehat{\Pi}^\rho|_{\delta\rho \rightarrow \rho \text{div}_\rho \mathbf{X}}. \quad (34)$$

where  $\text{div}_\rho \mathbf{X}$  is the divergence of  $\mathbf{X}$  with respect to  $\rho$ , see equation (7): If  $\rho = \rho(t)$  is one-parametric family then  $\rho \text{div}_\rho \mathbf{X} = \dot{\rho}_t|_{t=0}$  and  $\delta\rho = \dot{\rho}_t|_{t=0} \delta t$ . This formula can be easily checked for the canonical lifting  $\widehat{P}_{\lambda_0}^\rho$  and hence for an arbitrary lifting (24) in the affine space

of regular sdiff  $(M)$ -equivariant liftings:

$$\widehat{\text{ad}}_{\mathbf{X}} \widehat{P}_{\lambda_0}^{\rho}(\Delta) = \text{ad}_{\mathbf{X}} \left( \rho^{\widehat{\lambda}-\lambda_0} \circ \Delta \circ \rho^{\lambda_0-\widehat{\lambda}} \right) - \rho^{\widehat{\lambda}-\lambda_0} \circ \text{ad}_{\mathbf{X}} \Delta \circ \rho^{\lambda_0-\widehat{\lambda}} = \left( \widehat{\lambda} - \lambda_0 \right) \left[ \text{div}_{\rho} \mathbf{X}, \widehat{P}_{\lambda_0}^{\rho}(\Delta) \right].$$

(Compare with equation (31).)

**Corollary 1.** *Let  $\widehat{\Pi} = \widehat{\Pi}_{\lambda_0}^{\rho, (n)}$  be a regular sdiff  $\rho(M)$ -lifting defined on  $\mathcal{D}_{\lambda_0}^{(n)}(M)$ . Let the map  $\widehat{\Pi}$  be diff  $(M)$ -equivariant at the given operator  $\Delta_0$  of order  $n$ :*

$$\forall \mathbf{X}, \widehat{\text{ad}}_{\mathbf{X}} \widehat{\Pi}(\Delta) \Big|_{\Delta=\Delta_0} = 0.$$

*Then in the case the dimension of the manifold  $M$  is greater than 2, the weight  $\lambda_0 \neq \frac{1}{2}$  and the lifting  $\widehat{\Pi}$  is the distinguished (anti)-self-adjoint map up to a vertical map (see equation (28)). In particular*

$$\widehat{\Delta}_0^* = (-1)^n \widehat{\Delta}_0 + \text{vertical operators.}$$

We use this corollary to study diff  $(M)$ -liftings for higher order operators. (Regular diff  $(M)$ -invariant pencil liftings for operators of order 1 were already described in section 2.1 and example 3.1).

**4.3. Regular diff  $(M)$ -liftings for second order operators.** First we show that for weights  $\lambda_0 \neq 0, 1/2, 1$  on manifolds of dimension  $\geq 3$  the self-adjoint canonical pencil lifting described in section 2.2 is unique, regular and diff  $(M)$ -equivariant lifting map on operators of order  $\leq 2$ . (Note that this lifting's uniqueness can be obtained using the isomorphisms obtained by Duval and Ovsienko [2], and our analysis gives a complementary geometric picture behind their results.) Also we will explain why there is no diff  $(M)$ -equivariant lifting for the exceptional weights 0, 1/2 and 1.

Recall that for  $\lambda_0 \neq 0, 1/2, 1$  one can consider the canonical self-adjoint lifting map on  $\mathcal{D}_{\lambda_0}^{(2)}(M)$  (15), which sends the operator  $\Delta = S^{ij}(x)\partial_i\partial_j + T^i(x)\partial_i + R(x) \in \mathcal{D}_{\lambda_0}^{(2)}(M)$  to the pencil

$$\widehat{\Pi}_{\text{can}}(\Delta) = S^{ij}\partial_i\partial_j + \partial_j S^{ij}\partial_i + (2\widehat{\lambda} - 1)\gamma^i\partial_i + \widehat{\lambda}\partial_i\gamma^i + \widehat{\lambda}(\widehat{\lambda} - 1)\theta, \quad (35)$$

where the upper connection  $\gamma^i$  and the Brans-Dicke function  $\theta$  are equal to

$$\gamma^i = \frac{T^i - \partial_j S^{ij}}{2\lambda_0 - 1}, \quad \theta = \frac{1}{\lambda_0(\lambda_0 - 1)} \left( R - \frac{\lambda_0(\partial_i T^i - \partial_i p_j S^{ij})}{2\lambda_0 - 1} \right). \quad (36)$$

The map (35) defines a regular diff  $(M)$ -equivariant lifting, and moreover this lifting is self-adjoint. (See equations (15), (17) and (16) in subsection 2.2).

Let  $\widehat{\Pi}$  be an arbitrary regular diff  $(M)$ -equivariant lifting defined on  $\mathcal{D}_{\lambda_0}^{(2)}(M)$ . We then already know that  $\lambda_0 \neq \frac{1}{2}$  (see corollary 1) and we will show that the lifting,  $\widehat{\Pi}$ , coincides with the canonical lifting  $\widehat{\Pi}_{\text{can}}$ . (if  $\dim M \geq 3$ .)

Choose an arbitrary volume form  $\rho$ . The lifting  $\widehat{\Pi}$  has to be a regular sdiff  $\rho(M)$ -lifting, since it is a diff  $(M)$ -lifting. Hence it follows from Corollary 1 that

$$\widehat{\Pi}(\Delta) = \underbrace{\frac{\widehat{\lambda} + \lambda_0 - 1}{2\lambda_0 - 1} \widehat{P}_{\lambda_0}^{\rho}(\Delta) + \frac{\lambda_0 - \widehat{\lambda}}{2\lambda_0 - 1} \left( \widehat{P}_{\lambda_0}^{\rho}(\Delta) \right)^*}_{\text{distinguished lifting } \widehat{\Pi}_{\lambda_0, \text{disting.}}^{\rho, (2)}(\Delta)} + \underbrace{C(\widehat{\lambda}) \widehat{P}_{\lambda_0}^{\rho}(\Delta)(1) + D(\widehat{\lambda}) \left( \widehat{P}_{\lambda_0}^{\rho}(\Delta) \right)^*(1)}_{\text{vertical map}}. \quad (37)$$

Using calculations in example 3.2 collect the terms in the right hand side of this expansion, which are proportional to  $\partial_i \Gamma^i(x)$  and to  $\Gamma^i(x) \Gamma_i(x)$ , where  $\Gamma_i(x) = -\partial_i \log \rho(x)$  is connection of volume form,  $\rho$  ( $\Gamma^i(x) = S^{ik}(x) \Gamma_k(x)$ ). We come to equations

$$\begin{cases} -\lambda_0 C(\widehat{\lambda}) + (\lambda_0 - 1) D(\widehat{\lambda}) = 0, & (\text{terms proportional to } \partial_i \Gamma^i(x)) \\ \lambda_0^2 C(\widehat{\lambda}) + (\lambda_0 - 1)^2 D(\widehat{\lambda}) = (\widehat{\lambda} - \lambda_0)(\widehat{\lambda} + \lambda_0 - 1), & (\text{terms proportional to } \Gamma^i(x) \Gamma_i(x)) \end{cases}$$

These conditions uniquely define polynomials  $C(\widehat{\lambda})$  and  $D(\widehat{\lambda})$ . Thus independence on volume form implies that only one sdiff  $\rho$ -map  $\widehat{\Pi}(\Delta)$  may be diff  $(M)$ -equivariant. This is the canonical map  $\widehat{\Pi}_{\text{can.}}$ . Hence  $\widehat{\Pi} = \widehat{\Pi}_{\text{can.}}$ .

**Remark 8.** We wish to calculate explicitly the vertical map in (37). It is uniquely defined since it is equal to the difference between the canonical map (35) and the distinguished map. If  $\Gamma_i = -\log \rho(x)$  is the connection associated to  $\rho$ , then using (20) one can see that

$$\underbrace{C(\widehat{\lambda}) \widehat{P}_{\lambda_0}^\rho(\Delta)(1) + D(\widehat{\lambda}) \left( \widehat{P}_{\lambda_0}^\rho(\Delta) \right)^*(1)}_{\text{vertical map}} = \widehat{\Pi}_{\text{can.}}(\Delta) - \widehat{\Pi}_{\lambda_0, \text{distinguish.}}^\rho(\Delta) =$$

$$(\widehat{\lambda}(\widehat{\lambda} - 1) - \lambda_0(\lambda_0 - 1)) (\theta - 2\Gamma_i \gamma^i + S^{ij} \Gamma_i \Gamma_j).$$

We come to the cocycle like object  $c_\rho(\Delta) = \theta(x) - 2\gamma^i(x) \Gamma_i(x) + S^{ij}(x) \Gamma_i(x) \Gamma_j(x)$ .

Now we consider the exceptional weights. First consider  $\lambda_0 = 1/2$ . For second order operators acting on half-densities their adjoint acts on half-densities too. If  $\Delta = \Delta^*$  is a self-adjoint operator then choosing an arbitrary upper connection  $\gamma^i$  one can consider different self-adjoint operator pencils passing through this operator  $\Delta$  (see in more details in [7]). On the other hand there is no regular diff  $(M)$ -lifting defined on operators acting on half-densities, not even on the subspace of self-adjoint operators. Indeed every diff  $(M)$ -lifting has to be a sdiff  $\rho(M)$ -lifting, but in this case the lifting essentially depends on the volume form (see Proposition 3).

Now consider the case  $\lambda_0 = 0$ . An arbitrary second order operator acting on functions  $\Delta = S^{ij} \partial_i \partial_j + T^i \partial_i + R$  is defined uniquely by a contravariant symmetric tensor field  $S^{ij}$ , an upper connection  $\gamma^i = \partial_r S^{ri} \partial_i - T^i$  and a scalar function  $R = \Delta(1)$ . An operator  $\Delta$  is the sum of a normalised operator  $\Delta_{\text{norm}} = S^{ij} \partial_i \partial_j + T^i \partial_i$ , ( $\Delta_{\text{norm}}(1) = 0$ ) and a scalar function  $R(x) = \Delta(1)$ . The diff  $(M)$ -module  $\mathcal{D}^{(2)}(M)$  of second order operators is the direct sum of the module of normalised operators and scalar functions.

If  $\theta$  is an arbitrary Brans-Dicke function corresponding to an upper connection  $\gamma^i$  (see Appendix A) then according to (15) one can consider the self-adjoint canonical operator pencil

$$\Pi_{0, \text{can}}(\Delta) = S^{ij} \partial_i \partial_j + \partial_r S^{ri} \partial_i + (2\widehat{\lambda} - 1) \gamma^i \partial_i + \widehat{\lambda} \partial_i \gamma^i + \widehat{\lambda}(\widehat{\lambda} - 1) \theta$$

passing through the normalised operator  $S^{ij} \partial_i \partial_j + (\partial_r S^{ri} \partial_i - \gamma^i) \partial_i$ . This formula does not define a linear map even on the submodule of normalised operators ( $\Pi_{0, \text{can}}(\Delta_1 + \Delta_2) \neq \Pi_{0, \text{can}}(\Delta_1) + \Pi_{0, \text{can}}(\Delta_2)$ ).

Choose an arbitrary volume form  $\rho$  and consider an arbitrary regular sdiff  $\rho(M)$ -lifting on  $\mathcal{D}^{(2)}(M)$ . We set  $\lambda_0 = 0$  in (37) and come to the linear map

$$\widehat{\Pi}^\rho(\Delta) = (1 - \widehat{\lambda}) \widehat{P}_{\lambda_0}^\rho(\Delta) + \widehat{\lambda} \left( \widehat{P}_{\lambda_0}^\rho(\Delta) \right)^* + C(\widehat{\lambda}) \widehat{P}_{\lambda_0}^\rho(\Delta)(1) + D(\widehat{\lambda}) \left( \widehat{P}_{\lambda_0}^\rho(\Delta) \right)^*(1).$$

Now let us see how this lifting depends on  $\rho$ . Using equation (20) for the canonical lifting (19) we have

$$\begin{aligned} \widehat{\Pi}^\rho(\Delta) &= S^{ij}\partial_i\partial_j + \partial_r S^{ri}\partial_i + (2\widehat{\lambda} - 1)\gamma^i\partial_i + \widehat{\lambda}\partial_i\gamma^i + \\ &+ \widehat{\lambda}(\widehat{\lambda} - 1)(2\gamma^i\Gamma_i - \Gamma^i\Gamma_i) + (1 + C(\widehat{\lambda}) + D(\widehat{\lambda}))R(x) + D(\widehat{\lambda})(\partial_i\gamma^i - \partial_i\Gamma^i - \gamma^i\Gamma_i + \Gamma^i\Gamma_i), \end{aligned} \quad (38)$$

where as usual  $\Gamma_i(x) = -\partial_i \log \rho(x)$  is the flat connection induced by the volume form. One can see from this equation that the map  $\widehat{\Pi}^\rho$  depends on the choice of a volume form via the flat connection  $\Gamma_i$  for arbitrary vertical operators  $C(\widehat{\lambda})$  and  $D(\widehat{\lambda})$ . (Terms proportional to  $\Gamma^i(x)\Gamma_i(x)$  vanish only if  $D(\widehat{\lambda}) \neq 0$ , and the terms proportional to  $\partial_i\Gamma^i(x)$ ,  $\partial_i\gamma^i(x)$  vanish if  $D(\widehat{\lambda}) \equiv 0$ .) This dependance still exists if we define the map only on normalised operators. This implies that there are no  $\text{diff}(M)$ -lifting maps on  $\mathcal{D}_0^{(2)}$ , nor its subspace of normalised operators.

Analogous arguments work on the space of operators on densities of weight  $\lambda_0 = 1$ , since these operators are conjugate to operators on functions.

The following "limit" construction may, however, be of interest

**Example 4.3.** For arbitrary small  $\lambda_0$  consider the composition of the canonical volume form lifting  $P_{\lambda_0,0}^\rho$ . We come to

$$\begin{aligned} \widehat{\Pi}_{\text{can}} \circ P_{\lambda_0,0}^\rho (S^{ij}(x)\partial_i\partial_j + T^i(x)\partial_i + R(x)) &= \\ = S^{ij}(x)\partial_i\partial_j + \partial_r S^{ri}(x)\partial_i + (2\widehat{\lambda} - 1)\gamma^i(x) + \widehat{\lambda}\partial_i\gamma^i(x) + \widehat{\lambda}(\widehat{\lambda} - 1)\theta(x), \end{aligned}$$

where

$$\begin{cases} \gamma^i = \gamma^i(\lambda_0) = \partial_r S^{ri}(x) - T^i(x) + O(\lambda_0), \\ \theta = \theta(\lambda_0) = -\frac{R(x)}{\lambda_0} + R(x) + \partial_i\gamma^i(x) - \partial_i\Gamma^i(x) + \gamma^i(x)\Gamma_i(x) + O(\lambda_0), \end{cases}$$

Taking the limit  $\lambda_0 \rightarrow 0$  we come to a self-adjoint  $\text{sdiff}_\rho(M)$ -lifting defined on the subspace of normalised operators ( $R = 0$ ):

$$\widehat{\Pi}(\Delta) = \lim_{\lambda_0 \rightarrow 0} \widehat{\Pi}_{\text{can}} \circ P_{\lambda_0,0}^\rho (\Delta) = S^{ij}(x)\partial_i\partial_j + \partial_r S^{ri}(x)\partial_i + (2\widehat{\lambda} - 1)\gamma^i(x) + \widehat{\lambda}\partial_i\gamma^i(x) + \widehat{\lambda}(\widehat{\lambda} - 1)\theta_\rho(x),$$

with  $\gamma^i(x) = \partial_r S^{ri}(x) - T^i(x)$  and  $\theta_\rho = \partial_i\gamma^i(x) - \partial_i\Gamma^i(x) + \gamma^i(x)\Gamma_i(x)$ . ( $\theta_\rho(x)$  is Brans-Dicke function corresponding to the upper connection  $\gamma^i$  (see Appendix A:  $\theta_\rho - 2\gamma^i\Gamma_i + \Gamma^i\Gamma_i = \text{div}_\rho(\gamma - \Gamma)$ .) This exceptional  $\text{sdiff}_\rho(M)$ -lifting is self-adjoint.

#### 4.4. Regular $\text{diff}(M)$ -liftings for operators of order $\geq 3$ .

**Corollary 2.** *If  $M$  is a manifold of dimension  $\geq 3$ , then for an arbitrary weight  $\lambda_0$ , regular  $\text{diff}(M)$ -equivariant liftings defined on  $\mathcal{D}_{\lambda_0}^{(n)}(M)$  do not exist if  $n \geq 3$ .*

*Proof.* Suppose that  $\widehat{\Pi}$  is a  $\text{diff}(M)$ -equivariant lifting defined on  $\mathcal{D}_{\lambda_0}^{(n)}(M)$ . Choose an arbitrary volume form on  $M$  then  $\widehat{\Pi}$  has to be  $\text{sdiff}_\rho(M)$ -equivariant lifting. Due to the proposition 3 and corollary 1 the lifting  $\widehat{\Pi}$  is, up to a vertical map, the distinguished lifting  $\widehat{\Pi}_{\text{disting}}$ . Consider the variation (33) of this map with respect to a variation of the volume form. This condition cannot be satisfied for an arbitrary operator of order  $n - 2$ . Hence we come to contradiction.

This result is known from the works [8] and [10]. However this does not exclude the existence of diff-lifting maps on some subspaces of differential operators of an arbitrary order  $n$ . Based on the results of corollary 1 consider the following example

**Example 4.4.** Consider  $M = \mathbb{R}^d$ ,  $d$ -dimensional affine space. (We suppose that all functions and densities are rapidly decreasing at infinity) Choose a volume form  $\rho$  such that in a given Cartesian coordinates it is the coordinate volume form  $\rho = |Dx|$ . (In arbitrary Cartesian coordinates  $x^{i'}$  it will be equal to  $\rho = C|Dx'|$ , where  $C$  is a constant).

Operators in  $\mathbb{R}^d$  can be identified with contravariant symmetric tensors. Consider the subspace of symmetric divergenceless contravariant symmetric tensors:

$$\partial_{i_k} S^{i_1 \dots i_k}(x) \equiv 0. \quad (39)$$

Pick an arbitrary  $\lambda_0 \neq 1/2$ . We denote by  $L_+$  the linear space of operators formed by symmetric contravariant divergenceless tensors of even rank, and by  $L_-$  the linear space of operators formed by symmetric contravariant divergenceless tensors of odd rank. (We assume that functions, tensors of rank 0 belong to  $L_+$  also.) Using equation (28) for the distinguished lifting, consider the following liftings:

$$\widehat{\Pi}_{\pm}^{\rho} = \frac{\widehat{\lambda} + \lambda_0 - 1}{2\lambda_0 - 1} \widehat{P}_{\lambda_0}^{\rho}(\Delta) \pm \frac{\lambda_0 - \widehat{\lambda}}{2\lambda_0 - 1} \left( \widehat{P}_{\lambda_0}^{\rho}(\Delta) \right)^*. \quad (40)$$

$\widehat{\Pi}_+$  is self-adjoint map (28) on the whole space of operators. It is distinguished lifting map of spaces  $\mathcal{D}_{\lambda_0}^{(n)}(\mathbb{R}^m)$  for even  $n$ . Respectively anti-self-adjoint map  $\widehat{\Pi}_-$  is the distinguished lifting for odd  $n$ . One can show that  $\widehat{\Pi}_+$  is diff-equivariant on the subspace  $L_+$ , and respectively  $\widehat{\Pi}_-$  is diff-equivariant lifting on the subspace  $L_-$ . This can be checked directly by applying equations (33). Namely notice that the canonical lifting  $\widehat{P}_{\lambda_0}^{\rho}$  (see equation (19)) is also self-adjoint on  $L_+$  and it is anti-self-adjoint on  $L_-$ . E.g. for  $\Delta = S^{ikm} \partial_i \partial_k \partial_m$ ,  $\widehat{P}_{\lambda_0}^{\rho}(\Delta) = S^{ikm} \partial_i \partial_k \partial_m$  and  $\left( \widehat{P}_{\lambda_0}^{\rho}(\Delta) \right)^* = -\partial_i \partial_k \partial_m (S^{ikm}) = -S^{ikm} \partial_i \partial_k \partial_m$  since condition (39) is obeyed (we work in Cartesian coordinates where  $\rho = |Dx|$ ). Hence the canonical lifting  $\widehat{P}_{\lambda_0}^{\rho}$  is equal to  $\widehat{\Pi}_+$  on  $L_+$ , and  $\widehat{P}_{\lambda_0}^{\rho}$  is equal to  $\widehat{\Pi}_-$  on  $L_-$ . It follows from equations (33) and (40) that infinitesimal variations with respect to the volume form of the liftings  $\widehat{\Pi}_+^{\rho}$  on  $L_+$ , and of  $\widehat{\Pi}_-^{\rho}$  on  $L_-$  vanish. Hence due to equation (34) the lifting  $\widehat{\Pi}_+^{\rho}$  is diff( $M$ )-equivariant on  $L_+$ , and respectively  $\widehat{\Pi}_-^{\rho}$  is diff( $M$ )-equivariant on the subspace  $L_-$ .

## 5. TAYLOR SERIES FOR OPERATORS ON ALGEBRA OF DENSITIES AND SELF-ADJOINT LIFTINGS.

**5.1. Taylor series.** Let  $\widehat{\Delta}$  be an arbitrary operator defined on the algebra of densities on a manifold  $M$  which is provided with a volume form  $\rho$ . Pick an arbitrary weight  $\lambda_0$ . Consider the restriction  $\Delta_0 = \Delta|_{\widehat{\lambda}=\lambda_0}$  of the operator  $\Delta$  on densities of weight  $\lambda_0$  and the canonical lifting (19) of this operator,  $\widehat{\Delta}_0 = \widehat{P}_{\lambda_0}^{\rho}(\Delta_0)$ . The operators  $\widehat{\Delta}$  and  $\widehat{\Delta}_0$  coincide at  $\widehat{\lambda} = \lambda_0$ , hence we have expansion

$$\widehat{\Delta} = \widehat{\Delta}_0 + (\widehat{\lambda} - \lambda_0) \widehat{\Delta}_{(1)}. \quad (41)$$

Using this consideration repeatedly we come to the expansion of this operator as a power series:

$$\begin{aligned}\widehat{\Delta} &= \widehat{\Delta}_{(0)} = \widehat{\Delta}_0 + (\widehat{\lambda} - \lambda_0)\widehat{\Delta}_{(1)} = \widehat{\Delta}_0 + (\widehat{\lambda} - \lambda_0)\widehat{\Delta}_1 + (\widehat{\lambda} - \lambda_0)^2\widehat{\Delta}_{(2)} = \dots \\ &= \sum_{k=0}^p (\widehat{\lambda} - \lambda_0)^k \widehat{\Delta}_k + (\widehat{\lambda} - \lambda_0)^{p+1} \widehat{\Delta}_{(p+1)} = \sum_{k=0}^n (\widehat{\lambda} - \lambda_0)^k \widehat{\Delta}_k,\end{aligned}\quad (42)$$

where  $n$  is the order of the initial operator  $\widehat{\Delta}$ .

Here  $\Delta_j$  is the restriction of an operator  $\widehat{\Delta}_{(j)}$  to the subspace  $\mathcal{F}_{\lambda_0}(M)$  and  $\widehat{\Delta}_j$  is the canonical pencil lifting (19) of  $\Delta_j$ :

$$\Delta_j = \widehat{\Delta}_{(j)}|_{\widehat{\lambda}=\lambda_0}, \quad \widehat{\Delta}_j = P_{\lambda_0}^p(\Delta_j), \quad j = 0, 1, 2, \dots$$

The operators  $\Delta_k$  and  $\widehat{\Delta}_k$  have order  $\leq n - k$  if  $\widehat{\Delta}$  has order  $\leq n$ .

Formula (42) is an invariant expression for the expansions (4), (5).

**5.2. Description of all (anti)-self-adjoint liftings.** Now we use the Taylor series expansion for writing down a formula for all (anti)-self-adjoint liftings of an arbitrary operator.

Pick an operator  $\Delta$  of order  $n$  acting on densities of weight  $\lambda_0$ ,  $\Delta \in \mathcal{D}_{\lambda_0}^{(n)}(M)$ . We find all (anti)-self-adjoint liftings of this operator, more precisely self-adjoint liftings if  $n$  is even and anti-self-adjoint liftings if  $n$  is odd (see remark 2).

Consider first the case of  $\lambda_0 = 1/2$ . In this case the operator  $\Delta^*$  acts on half-densities also, so a (anti)-self-adjoint-lifting is possible iff  $\Delta^* = (-1)^n \Delta$ , where  $n$  is the order of  $\Delta$ . Let  $\widehat{\Delta}$  be such a lifting of  $\Delta = \Delta_0$ :  $\widehat{\Delta}|_{\widehat{\lambda}=\lambda_0} = \Delta_0$  and  $\widehat{\Delta}^* = (-1)^n \widehat{\Delta}$ . Consider the Taylor series expansion (42) of the operator  $\widehat{\Delta}$ :  $\widehat{\Delta} = \sum_{k=0}^n (\widehat{\lambda} - 1/2)^k \widehat{\Delta}_k$ . Note that canonical liftings preserve self-adjointness (see (21)). Recalling also that the operator  $\widehat{\lambda} - 1/2$  is anti-self-adjoint:  $(\widehat{\lambda} - 1/2)^* = -(\widehat{\lambda} - 1/2)$  we come from this expansion to the fact that the operator  $\widehat{\Delta}$  is self-adjoint (anti-self-adjoint) lifting of  $\Delta = \Delta_0$  if and only if all the operators  $\{\Delta_0, \Delta_2, \dots, \Delta_{2k}, \dots\}$  are self-adjoint (anti-self-adjoint) and all the operators  $\{\Delta_1, \Delta_3, \dots, \Delta_{2k+1}, \dots\}$  are anti-self-adjoint (self-adjoint).

Thus we have described all (anti)-self-adjoint liftings for arbitrary operator acting on half-densities.

**Remark 9.** Recall that in the previous section we showed that in general there are no regular self-adjoint diff( $M$ )-lifting maps on the space  $\mathcal{D}_{1/2}^{(n)}(M)$ . Here we considered self-adjoint liftings for individual operators  $\Delta \in \mathcal{D}_{1/2}^{(n)}(M)$ .

Now we consider the case of lifting of operators acting on densities of an arbitrary weight  $\lambda_0 \neq \frac{1}{2}$ . Let  $\widehat{\Delta}$  be a lifting of  $\Delta = \Delta_0 \in \mathcal{D}_{\lambda_0}^{(n)}(M)$ ,  $\lambda_0 \neq 1/2$ . Due to (41)

$$\widehat{\Delta} = \widehat{\Delta}_0 + (\widehat{\lambda} - \lambda_0)\widehat{\Delta}_{(1)},$$

where as usual  $\widehat{\Delta}_0$  is canonical lifting of operator  $\Delta_0 = \widehat{\Delta}|_{\widehat{\lambda}=\lambda_0}$ . To find conditions on the operator  $\widehat{\Delta}_{(1)}$  such that  $\widehat{\Delta}$  is (anti)-self-adjoint it is convenient to consider its Taylor series

expansion (42) around  $\lambda_0' = 1/2$ :  $\widehat{\Delta} = \widehat{\Delta}_0 + (\widehat{\lambda} - \lambda_0)\widehat{\Delta}_{(1)} =$

$$\widehat{\Delta}_0 + (\widehat{\lambda} - \lambda_0) \left[ \widehat{\Delta}_1 + (\widehat{\lambda} - 1/2) \widehat{\Delta}_2 + (\widehat{\lambda} - 1/2)^2 \widehat{\Delta}_3 + \cdots + (\widehat{\lambda} - 1/2)^{n-1} \widehat{\Delta}_n \right] \quad (43)$$

As usual here  $\Delta_i$  is an operator of order  $\leq n - i$  and  $\widehat{\Delta}_i$  is the canonical lifting of  $\Delta_i$ :  $\widehat{\Delta}_i = \widehat{P}_{\lambda_0}^{\rho}(\Delta_i)$ . Take the adjoint of this expansion, and using the fact that  $(\widehat{\lambda} - 1/2)$  is anti-self-adjoint we compare the terms proportional to the powers  $(\widehat{\lambda} - 1/2)^k$ . We see that the condition of (anti)-self-adjointness for  $\widehat{\Delta}$ ,  $\widehat{\Delta}^* = (-1)^n \widehat{\Delta}$  is equivalent to the equations:

$$\Delta_k - (-1)^{n-k} \Delta_k^* = \left( \lambda_0 - \frac{1}{2} \right) (\Delta_{k+1} - (-1)^{n-k} \Delta_{k+1}^*), \quad (k = 0, 1, 2, \dots, n-1)$$

for the operators  $\Delta_k$ ,  $k = 0, 1, 2, \dots$ , the ‘‘coefficients’’ in the power series (43), (The last operator  $\Delta_n$  is just a function.) One can parametrise solutions of this equation in the following way: For a given operator  $\Delta_0 \in \mathcal{D}_{\lambda_0}^{(n)}(M)$  pick arbitrary operators standing on even places, operators  $\{\Delta_2, \Delta_4, \dots\}$  such that  $\Delta_{2k} \in \mathcal{D}_{\lambda_0}^{(n-2k)}$  ( $k = 1, 2, \dots$ ). Then the operators  $\{\Delta_1, \Delta_3, \dots\}$  at the odd places are equal to

$$\Delta_{2k+1} = \frac{\Delta_{2k} - (-1)^n \Delta_{2k}^*}{2\lambda_0 - 1} + (2\lambda_0 - 1) \frac{\Delta_{2k+2} + (-1)^n \Delta_{2k+2}^*}{4}, \quad k = 0, 2, 4, \dots \quad (44)$$

In particular we see that all operators  $\Delta_k$  have order  $\leq n - k$ .

Note that in the special case if the defining operators  $\{\Delta_2, \Delta_4, \dots\}$  vanish then for all  $k \geq 1$  the operators  $\Delta_k$  vanish, and formula (43) reduces to the distinguished map (28).

**Remark 10.** The constructions above fixes the one-to-one correspondence between the space of all self-adjoint operator pencils passing through given operators and the space of sequences of operators  $\{\Delta_2, \Delta_4, \dots\}$ . Both these spaces are independent of the volume form, but the correspondence established by the formulae above depends on the choice of the volume form which defines the Taylor series expansion (44).

**Example 5.1.**  $n = 2$ . Self-adjoint liftings are parameterised by just one operator of order 0, i.e. a function:  $\Delta_2 = F(x)$ . Pick an arbitrary operator  $\Delta = \Delta_0 \in \mathcal{D}_{\lambda_0}^{(2)}$  ( $\lambda_0 \neq 1/2$ ) and consider

$$\widehat{\Delta} = \widehat{\Delta}_0 + (\widehat{\lambda} - \lambda_0) \left( \widehat{\Delta}_1 + (\widehat{\lambda} - 1/2)F \right),$$

where due to (44),  $F$  is an arbitrary function,  $\Delta_1 = \frac{\Delta_0 - \Delta_0^*}{2\lambda_0 - 1} + (\lambda_0 - \frac{1}{2})F$ ,  $\widehat{\Delta}_0 = \widehat{P}_{\lambda_0}^{\rho}(\Delta_0)$  and  $\widehat{\Delta}_1 = \widehat{P}_{\lambda_0}^{\rho}(\Delta_1)$ . This expansion gives all operator pencils  $\widehat{\Delta}$  passing through given operator  $\Delta_0$  (as always  $\{\Delta_{\lambda}\}$ :  $\Delta_{\lambda} = \widehat{\Delta}|_{\widehat{\lambda}=\lambda}$ ). The family of these operators is in one-to-one correspondence with functions  $F$  on  $M$ . This correspondence depends on volume form <sup>2</sup>.

Now consider anti-self-adjoint liftings for third order operators.

<sup>2</sup>Compare with expansion (15) in section 2.1 where, for weights  $\lambda_0 \neq 0, 1$ , the space of self-adjoint operators passing through a given operator was parametrised by the function  $F = \widehat{\Delta}(1)$  independent of the choice of volume form (see also end of section 4.3).

**Example 5.2.**  $n = 3$ . Pick an arbitrary third order operator  $\Delta_0 \in \mathcal{D}_{\lambda_0}^{(3)}$ ,  $\lambda_0 \neq 1/2$ . An antiself-adjoint lifting has the form

$$\widehat{\Delta} = \widehat{\Delta}_0 + (\widehat{\lambda} - \lambda_0) \left( \widehat{\Delta}_1 + (\widehat{\lambda} - 1/2)\widehat{\Delta}_2 + (\widehat{\lambda} - 1/2)^2\widehat{\Delta}_3 \right),$$

where  $\Delta_2$  is an arbitrary first order operator acting on half-densities, and the operators  $\Delta_1$  and  $\Delta_3$  (which is a function) are expressed through the operator  $\Delta_2$  and the initial operator  $\Delta_0$ . Recalling that by (11)  $\Delta_2$  can be represented as  $\Delta_2 = \mathcal{L}_A + S(x)$ , where  $A$  is a vector field and  $S(x)$  is a scalar, and using equations (44), we come to

$$\widehat{\Delta} = \frac{\widehat{\Delta}_0 - \widehat{\Delta}_0^*}{2} + \frac{2\widehat{\lambda} - 1}{2\lambda_0 - 1} \frac{\widehat{\Delta}_0 + \widehat{\Delta}_0^*}{2} + \left( \widehat{\lambda}(\widehat{\lambda} - 1) - \lambda_0(\lambda_0 - 1) \right) \left( \widehat{\mathcal{L}}_A + \frac{2\widehat{\lambda} - 1}{2\lambda_0 - 1} S(x) \right), \quad (45)$$

where  $\widehat{\Delta}_0$ ,  $\widehat{\Delta}_0^*$  and  $(\widehat{\mathcal{L}}_A)$  are the canonical liftings (19) of the initial operator  $\Delta_0$ , its adjoint, and of the Lie derivative:

$$\widehat{\mathcal{L}}_A = P_{\lambda_0}^{\rho}((\mathcal{L}_A)) = \widehat{\mathcal{L}}_A - (\widehat{\lambda} - \lambda_0)\text{div}_{\rho} \mathbf{A},$$

where  $\widehat{\mathcal{L}}_A$  is the Lie derivative on the algebra of densities (see equation (10)). Equations (45) describe all anti-self-adjoint liftings of third order operator in terms of arbitrary vector field and scalar function. They contain interesting geometrical data which we will discuss elsewhere.

## 6. STRICTLY REGULAR PROJ-LIFTINGS AND SELF-ADJOINT LIFTINGS

In section 4 we studied regular sdiff  $\rho$ -lifting maps and studied the special role of distinguished (anti)-self-adjoint lifting (28). In the previous section we studied all self-adjoint pencils passing through any given operator  $\Delta$ . We used for this the Taylor series expansion based on strictly regular sdiff  $\rho$ -equivariant canonical liftings (19). In section 4 we considered (anti)-self-adjoint liftings such that on the whole space  $\mathcal{D}_{\lambda_0}^{(n)}(M)$ , they are regular sdiff-equivariant lifting maps. In this section we will give a short description of another construction of self-adjoint liftings. We will construct self-adjoint liftings which are linear maps on  $\mathcal{D}_{\lambda_0}^{(n)}$  equivariant with respect to a smaller algebra, the algebra of projective transformations. For simplicity we consider here the case if  $M$  is  $d$ -dimensional affine space,  $M = \mathbb{R}^d$ . The algebra  $\text{proj}(\mathbb{R}^d)$  is a finite dimensional subalgebra of vector fields on  $\mathbb{R}^d$ . This algebra corresponds to the group of projective transformations of  $\mathbb{R}P^d = \text{SL}(d+1, \mathbb{R})$ , the group of linear unimodular transformations of  $\mathbb{R}^{d+1}$ . The algebra  $\text{proj}(\mathbb{R}^d)$  is generated by translations  $\partial_i$ , linear transformations  $x^i \partial_k$ , and special projective transformations  $x^i x^k \partial_k$ , ( $i, k = 1, \dots, d$ ). Its dimension is equal to  $d + d^2 + d = (d+1)^2 - 1$ .

Pick an arbitrary  $\lambda_0$ , we will describe the construction of a strictly regular  $\text{proj}(\mathbb{R}^d)$ -equivariant pencil map on  $\mathcal{D}_{\lambda_0}(\mathbb{R}^d)$ . This construction is based on the classical results of Lecomte and Ovsienko of constructing a  $\text{proj}$ -equivariant full symbol map on differential operators (see [9] or the book [11]).

Symbols of differential operators are contravariant symmetric tensor fields on  $\mathbb{R}^d$  which can be identified with functions on the cotangent bundle  $T^*\mathbb{R}^d$  which are polynomials on the fibres. In particular to an arbitrary operator  $\Delta = S^{i_1 \dots i_n} \partial_{i_1 \dots i_n} + \dots$  one can assign its principal symbol, the contravariant symmetric tensor field  $[\Delta] = S^{i_1 \dots i_n} \xi_1 \xi_2 \dots \xi_n$  ( $\xi_i$  are the



standard coordinates in the fibres of the cotangent bundle  $T^*\mathbb{R}^d$ ). This assignment defines a canonical  $\text{Diff}(\mathbb{R}^d)$ -equivariant map from  $\mathcal{D}_\lambda^{(n)}(M)$  to the space of symbols. However modules of differential operators and modules of symbols are not  $\text{diff}$ -isomorphic. Lecomte and Ovsienko constructed isomorphism, full symbol map,  $\sigma_\lambda$ , such that for any given weight  $\lambda$  it is a  $\text{proj}$ -equivariant isomorphism, between the space of linear differential operators on densities, of a given weight  $\lambda$ , and symbols of operators, i.e. the space of contravariant symmetric tensor fields on  $\mathbb{R}^d$ . This  $\text{proj}$ -equivariant isomorphism is uniquely defined by the normalisation condition of preserving the principal symbol of operator:

$$\sigma_\lambda (S^{i_1 \dots i_n} \partial_1 \dots \partial_n) = S^{i_1 \dots i_n} \xi_1 \dots \xi_n + \dots \quad (46)$$

(Here and later we will identify contravariant symmetric tensor fields with functions polynomial on the fibers) For an arbitrary operator  $\Delta = \sum_n L^{i_1 \dots i_n} \partial_{i_1} \dots \partial_{i_n} \in \mathcal{D}_{\lambda_0}(\mathbb{R}^d)$

$$\sigma_\lambda (\Delta) = \sum_n \left( \sum_{k=0}^n c_k^{(n)}(\lambda) \partial_{p_1} \dots \partial_{p_k} L^{p_1 \dots p_k i_1 \dots i_{n-k}} \xi_{i_1} \dots \xi_{i_{n-k}} \right), \quad (47)$$

where the coefficients  $c_k^{(n)}(\lambda)$  are polynomials in  $\lambda$ , of degree  $\leq n - k$ ,  $c_0^{(n)} = 1$  (normalisation condition (46)). Equivariance of the isomorphism  $\sigma_\lambda$  with respect to the algebra  $\text{proj}(\mathbb{R}^d)$  and the normalisation conditions uniquely define these polynomials. Namely consider the equivariance condition  $\mathcal{L}_{\mathbf{X}} \sigma_\lambda (\Delta) = \sigma_\lambda (\text{ad}_{\mathbf{X}} \Delta)$ . These conditions are evidently obeyed for translations and infinitesimal affine transformations (the vector fields  $\partial_i$  and  $x^i \partial_k$ ). Considering equivariance for the special projective transformations,  $\mathbf{X}^{(i)} = x^i x^k \partial_k$ <sup>3</sup> and applying these conditions to the full symbol map (47) one comes to recurrent relations which define the polynomials  $c_k^{(n)}(\lambda)$  via the polynomials  $c_{k-1}^{(n-1)}(\lambda_0)$ . One finds that

$$c_k^{(n)} = (-1)^k \frac{\binom{n}{k} \binom{\lambda(d+1) + n - 1}{k}}{\binom{2n - k + d}{k}}, \quad (48)$$

where  $\binom{a}{m}$  are the binomial coefficients. (See for detail [9] or book [11].)

One can consider the map  $Q_\lambda$ , which is inverse to the full symbol map, a so called quantisation map (see [9] and [11]). It follows from normalisation condition (46), equations (47) and (48) that for the quantisation map

$$Q_\lambda (L^{i_1 \dots i_n} \xi_1 \dots \xi_n) = L^{i_1 \dots i_n} \partial_1 \dots \partial_n + \dots = \sum_{k=0}^n \tilde{c}_k^{(n)}(\lambda) \partial_{p_1} \dots \partial_{p_k} L^{p_1 \dots p_k i_1 \dots i_{n-k}} \partial_{i_1} \dots \partial_{i_{n-k}},$$

where the  $\tilde{c}_k^{(n)}(\hat{\lambda})$  are polynomials in  $\lambda$  of order  $n - k$ , which can be recurrently obtained from polynomials  $c_k^{(n)}(\hat{\lambda})$  in equation (48),  $c_0^{(n)} = 1$ ,  $c_1^{(n)} = -c_1^{(n-1)}, \dots$

<sup>3</sup> $\text{ad}_{\mathbf{X}^{(i)}} \Delta = (\mathcal{L}_{\mathbf{X}^{(i)}} L)^{i_1 \dots i_n} \partial_{i_1} \dots \partial_{i_n} - \mathcal{P}_{pq}^{(i) i_1}(\lambda_0, n) S^{p q i_2 \dots i_{n-1}} \partial_{i_1} \dots \partial_{i_{n-1}}$ , where

$$\mathcal{P}_{pq}^{(i) i_1}(\lambda_0, n) = n \left( \frac{n-1}{2} (\delta_p^i \delta_q^{i_1} + \delta_q^i \delta_p^{i_1}) + \lambda(d+1) \delta_p^i \delta_q^{i_1} \right)$$

**Example 6.1.** Consider the full symbol map and its inverse on  $\mathcal{D}_\lambda^{(2)}(\mathbb{R})$ : Given  $\Delta \in \mathcal{D}_\lambda^{(2)}(\mathbb{R})$ ,  $\Delta = a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} + c(x)$

$$\begin{aligned} \sigma_\lambda(\Delta) &= \sigma_\lambda\left(a(x)\frac{d^2}{dx^2}\right) + \sigma_\lambda\left(b(x)\frac{d}{dx}\right) + \sigma_\lambda(c(x)) = \\ &\left(a(x)\xi^2 - \frac{2\lambda+1}{2}a_x(x)\xi + \frac{\lambda(2\lambda+1)}{3}a_{xx}(x)\right) + (b(x)\xi - \lambda b_x(x)) + c(x). \end{aligned}$$

Respectively for the quantisation map  $Q_\lambda = \sigma_\lambda^{-1}$  we have

$$\begin{aligned} Q_\lambda(a(x)\xi^2 + b(x)\xi + c(x)) &= Q_\lambda(a(x)\xi^2) + Q_\lambda(b(x)\xi) + Q_\lambda(c(x)) = \\ &\left(a(x)\frac{d^2}{dx^2} + \frac{2\lambda+1}{2}a_x(x)\frac{d}{dx} + \frac{\lambda(2\lambda+1)}{6}a_{xx}(x)\right) + \left(b(x)\frac{d}{dx} + \lambda b_x(x)\right) + c(x). \end{aligned}$$

Now using the  $\text{proj}(\mathbb{R}^n)$ -equivariant full symbol map,  $\sigma_\lambda$ , and its inverse, the quantisation map  $Q_\lambda$  we construct strictly regular  $\text{proj}$ -equivariant pencil liftings.

Pick an arbitrary weight  $\lambda = \lambda_0$  and consider on the  $\mathcal{D}_{\lambda_0}(\mathbb{R}^n)$  a map

$$\mathcal{D}_{\lambda_0}(\mathbb{R}^n) \ni \Delta \mapsto \widehat{\Delta} = \widehat{\Pi}(\Delta): \quad \widehat{\Pi}(\Delta)|_{\widehat{\lambda}=\lambda} = Q_\lambda(\sigma_{\lambda_0}(\Delta)).$$

With some abuse of language we say that

$$\widehat{\Pi}(\Delta)|_{\widehat{\lambda}=\lambda} = Q_{\widehat{\lambda}}(\sigma_{\lambda_0}(\Delta)). \quad (49)$$

The map  $\widehat{\Pi}$  is a composition of  $\text{proj}(\mathbb{R}^d)$ -equivariant maps,  $\widehat{\Pi}_{\widehat{\lambda}=\lambda_0} = Q_{\lambda_0} \circ \sigma_{\lambda_0} = \text{id}$  and it preserves the order of operators. Hence the map  $\widehat{\Pi}$  is a strictly regular  $\text{proj}$ -equivariant lifting of  $\mathcal{D}_{\lambda_0}(\mathbb{R}^d)$ . This is a unique lifting, by the uniqueness of map (47).

Let us look in more detail at the structure of the  $\text{proj}$ -equivariant lifting map  $\widehat{\Pi}$ . Pick  $n$  and consider the restriction of this strictly regular lifting map to the subspace of operators of order  $\leq n$ .

For any operator  $\Delta = \Delta_{(0)} \in \mathcal{D}_{\lambda_0}^{(n)}(\mathbb{R}^d)$  consider an operator  $\Delta_0$  with the same principal symbol,  $[\Delta] = [\Delta_0]$  such that the value of the full symbol map  $\sigma_{\lambda_0}$  at  $\Delta_0$  is equal to its principal symbol.

$$\Delta_0 = \text{pr}_0(\Delta) = Q_{\lambda_0}([\Delta]).$$

The lifting map  $\widehat{\Pi}_0 = \widehat{\Pi} \circ \text{pr}_0$  maps an operator  $\Delta$  to an operator  $\widehat{\Delta}_0 = Q_{\widehat{\lambda}}([\Delta])$  such that for the corresponding operator pencil  $\{\Delta_\lambda\}$  ( $\Delta_\lambda = Q_\lambda([\Delta])$ ) all operators  $\Delta_\lambda$  have the same symbol as the operator  $\Delta$ . The operator  $\Delta_{(1)} = \Delta - \Delta_0$  is an operator of order  $\leq n$ ,  $\Delta_{(1)} \in \mathcal{D}_{\lambda_0}^{(n-1)}$ . Applying the same procedure to the operator  $\Delta_{(1)}$  and using these considerations repeatedly we come to a  $\text{proj}$ -equivariant decomposition of operators in  $\mathcal{D}_{\lambda_0}^{(n)}(\mathbb{R}^n)$ , and the lifting map (49) acts on the  $n+1$  components individually:

$$\begin{aligned} \Delta = \Delta_{(0)} &= \Delta_0 + \Delta_1 + \cdots + \Delta_n, \quad , \Delta_{(i)} = \Delta_i + \cdots + \Delta_n, \\ \widehat{\Pi} &= \widehat{\Pi}_0 + \widehat{\Pi}_1 + \cdots + \widehat{\Pi}_n, \quad , \widehat{\Pi}_k = \widehat{\Pi} \circ \text{pr}_k, \end{aligned} \quad (50)$$

where  $\Delta_i = \text{pr}_i(\Delta) = Q_{\lambda_0}([\Delta_{(i)}])$ . The operators  $\Delta_i$  belong to  $\mathcal{D}_{\lambda_0}^{(n-i)}(\mathbb{R}^d)$ , and the operator pencil  $\Pi_i(\Delta) = \Pi_i(\Delta_i)$  is a lifting of the operator  $\Delta_i$ . All operators of this pencil have the same symbol. The decomposition (50) is a tool to describe all regular and all self-adjoint regular pencil liftings which are equivariant with respect to the algebra  $\text{proj}(\mathbb{R}^d)$ .

Let  $\Pi$  be an arbitrary proj-equivariant regular lifting of  $\mathcal{D}_{\lambda_0}^{(n)}(\mathbb{R}^d)$ . Then it follows from uniqueness arguments that

$$\Pi(\Delta) = \sum_{k=0}^n P_k(\widehat{\lambda}) \widehat{\Pi}_k(\Delta), \quad (51)$$

where  $P_k(\widehat{\lambda})$  ( $k = 0, 1, 2, \dots, n$ ) are arbitrary polynomials on  $\widehat{\lambda}$  of order  $n - k$  which obey the conditions that any polynomial  $P_k(\widehat{\lambda})$  has an order  $\leq k$  and  $P_k(\widehat{\lambda})|_{\widehat{\lambda}=\lambda_0} = 1$ , since the lifting is regular and  $\Pi(\Delta)|_{\widehat{\lambda}=\lambda_0} = \Delta$ . It follows from these conditions that

$$P_0(\Delta) = 1, P_1(\widehat{\lambda}) = 1 + c(\widehat{\lambda} - \lambda_0), \quad \text{and in general } P_k(\widehat{\lambda}) = 1 + (\widehat{\lambda} - \lambda_0)G_{k-1}(\widehat{\lambda}), \quad (52)$$

where  $G_{k-1}(\widehat{\lambda})$  is an arbitrary polynomial of order  $\leq k - 1$ . We see that the space of liftings is a  $1 + \dots + n = \frac{n(n+1)}{2}$ -dimensional affine space. (Compare with the dimension of the space of regular sdiff  $\rho$ -liftings in Proposition 1).

The same uniqueness arguments imply that in the decomposition (50) all operators  $\widehat{\Delta}_i = \widehat{\Pi}_i \widehat{\lambda}(\Delta_i)$  are self-adjoint or anti-self-adjoint:  $(\widehat{\Delta}_i)^* = (-1)^{n-i} \widehat{\Delta}_i$ . Hence the regular lifting (51) is a self-adjoint proj-equivariant regular lifting if  $n$  is even (respectively anti-self-adjoint lifting if  $n$  is odd) in the case that the polynomials  $P_k(\widehat{\lambda})$  obey the additional condition of self-adjointness:

$$(P_k(\widehat{\lambda}))^* = (-1)^k P_k(\widehat{\lambda}).$$

These conditions with the conditions from equation (52) imply that

$$P_0(\Delta) = 1, P_1(\widehat{\lambda}) = \frac{2\widehat{\lambda} - 1}{2\lambda_0 - 1}, \quad P_2(\widehat{\lambda}) = 1 + b \left( \widehat{\lambda} (\widehat{\lambda} - 1) - \lambda_0 (\lambda_0 - 1) \right)$$

and in general  $P_{2k}(\widehat{\lambda}) = 1 + \sum_{r=1}^k c_r \left( t^{2r}(\widehat{\lambda}) - t^{2r}(\lambda_0) \right)$  and

$$P_{2k+1}(\widehat{\lambda}) = \frac{t(\widehat{\lambda})}{t(\lambda_0)} \left( 1 + \sum_{r=1}^k d_r \left( t^{2r}(\widehat{\lambda}) - t^{2r}(\lambda_0) \right) \right), \quad \left( \lambda_0 \neq \frac{1}{2} \right), \quad (53)$$

where  $c_i, d_j$  are constant coefficients,  $t(\widehat{\lambda}) = \widehat{\lambda} - \frac{1}{2}$  is anti-self-adjoint linear polynomial in  $\widehat{\lambda}$ :  $t^*(\widehat{\lambda}) = \left( \widehat{\lambda} - \frac{1}{2} \right)^* = -\widehat{\lambda} + \frac{1}{2} = -t(\widehat{\lambda})$ . We see that space of liftings is  $\frac{n^2 - p(n)}{4}$ -dimensional affine space, here  $p(n) = 0$  for even  $n$  and  $p(n) = 1$  for odd  $n$ . (Compare with the dimension of the space of self-adjoint regular sdiff  $\rho$ -liftings, see proposition(2).)

**Example 6.2.** Consider liftings on second order operators acting on  $\mathbb{R}$ . We already calculated the full symbol map and the quantisation map in example 6.1. We have that for  $\Delta = a(x) \frac{d^2}{dx^2} + b(x) \frac{d}{dx} + c(x)$  acting on densities of weight  $\lambda_0$ ,  $\widehat{\Pi}(\Delta) = Q_{\widehat{\lambda}} \circ \sigma_{\lambda_0}(\Delta)$ .

Consider the decomposition (50). The principal symbol of the operator  $\Delta$  is equal to  $[\Delta] = a\xi^2$ , hence (see example 6.1)

$$\Delta_0 = \text{pr}_0(\Delta) = Q_{\lambda_0}(a\xi^2) = a(x) \frac{d^2}{dx^2} + \frac{2\lambda_0 + 1}{2} a_x \frac{d}{dx} + \frac{\lambda_0(2\lambda_0 + 1)}{6} a_{xx}$$

and

$$\widehat{\Delta}_0 = \widehat{\Pi}_0(\Delta) = \widehat{\Pi}(\text{pr}_0(\Delta)) = Q_{\widehat{\lambda}}([\Delta]) = Q_{\widehat{\lambda}}(a\xi^2) = a(x)\partial_x^2 + \frac{2\widehat{\lambda}+1}{2}a_x\partial_x + \frac{\widehat{\lambda}(2\widehat{\lambda}+1)}{6}a_{xx}.$$

(Note that for differential operators in the algebra of densities on  $\mathbb{R}$  we have partial derivatives  $\partial_x$ , not  $\frac{d}{dx}$ .) Respectively  $\Delta_{(1)} = \Delta - \Delta_0 = \left(b - \frac{2\lambda_0+1}{2}a_x\right) d/dx + \left(c - \frac{\lambda_0(2\lambda_0+1)}{6}a_{xx}\right)$ ,

$$\Delta_1 = \text{pr}_1(\Delta) = Q_{\lambda_0}(\Delta_{(1)}) = \left(b - \frac{2\lambda_0+1}{2}a_x\right) \frac{d}{dx} + \lambda_0 \left(b_x - \frac{2\lambda_0+1}{2}a_{xx}\right),$$

$$\widehat{\Delta}_1 = \widehat{\Pi}_1(\Delta_1) = \widehat{\Pi}(\text{pr}_1(\Delta)) = Q_{\widehat{\lambda}}([\Delta_{(1)}]) = \left(b - \frac{2\lambda_0+1}{2}a_x\right) \partial_x + \widehat{\lambda} \left(b_x - \frac{2\lambda_0+1}{2}a_{xx}\right),$$

and  $\Delta_2 = \Delta - \Delta_0 - \Delta_1 = c - \lambda_0 b_x + \frac{\lambda_0(2\lambda_0+1)}{3}$ ,  $\widehat{\Delta}_2 = \widehat{\Pi}_2(\Delta) = c - \lambda_0 b_x + \frac{\lambda_0(2\lambda_0+1)}{3}$ .

We have the decomposition of the strictly regular pencil:

$$\widehat{\Delta} = \widehat{\Pi}(\Delta) = \widehat{\Pi}_0(\Delta) + \widehat{\Pi}_1(\Delta) + \widehat{\Pi}_2(\Delta) = \widehat{\Delta}_0 + \widehat{\Delta}_1 + \widehat{\Delta}_2.$$

All regular proj-equivariant pencil maps on  $\mathcal{D}_{\lambda_0}^{(2)}(\mathbb{R})$  according to (51) are of the form

$$\widehat{\Pi}(\Delta) = \widehat{\Delta}_0 + \left(1 + k_1(\widehat{\lambda} - \lambda_0)\right) \widehat{\Delta}_1 + \left(1 + k_2(\widehat{\lambda} - \lambda_0) + k_3(\widehat{\lambda} - \lambda_0)^2\right) \widehat{\Delta}_2,$$

where  $k_1, k_2, k_3$  are constants. This is a 3-dimensional affine plane of liftings. All regular self-adjoint proj-equivariant pencil maps on  $\mathcal{D}_{\lambda_0}^{(2)}(\mathbb{R})$  according to (53) have the form

$$\widehat{\Pi}(\Delta) = \widehat{\Delta}_0 + \left(\frac{2\widehat{\lambda}-1}{2\lambda_0-1}\right) \widehat{\Delta}_1 + \left(1 + k \left(\widehat{\lambda}(\widehat{\lambda}-1) - \lambda_0(\lambda_0-1)\right)\right) \widehat{\Delta}_2, \quad (54)$$

where  $k$  is a constant.

This is an affine line of liftings. Compare these liftings with the canonical self-adjoint lifting (35), which in this case has the following appearance:

$$\widehat{\Pi}_{\text{can}}(\Delta) = a(x)\partial_x^2 + a_x(x)\partial_x + (2\widehat{\lambda}-1)\gamma(x)\partial_x + \widehat{\lambda}\partial_x\gamma + \widehat{\lambda}(\widehat{\lambda}-1)\theta(x), \quad (55)$$

where the upper connection  $\gamma$  and the Brans-Dicke function  $\theta$  are equal to

$$\gamma(x) = \frac{b(x) - a_x(x)}{2\lambda_0 - 1}, \quad \theta(x) = \frac{1}{\lambda_0(\lambda_0 - 1)} \left(c(x) - \frac{\lambda_0(b_x(x) - a_{xx}(x))}{2\lambda_0 - 1}\right), \quad (56)$$

( $\lambda_0 \neq 0, 1/2, 1$ ). Now substituting (56) into (54) and comparing with (55), we find

$$\widehat{\Pi}_{\kappa}(\Delta) = \widehat{\Pi}_{\text{can}}(\Delta) + \kappa \left(\widehat{\lambda}(\widehat{\lambda}-1) - \lambda_0(\lambda_0-1)\right) \left(\theta(x) - 2\gamma_x(x) + \frac{2}{3}a_{xx}(x)\right), \quad (57)$$

( $\kappa = \lambda_0(\lambda_0 - 1)k - 1$ ). The pencil maps  $\widehat{\Pi}(\Delta)$  are regular and proj-equivariant, whilst  $\widehat{\Pi}_{\text{can}}(\Delta)$  is a regular diff-equivariant map. Hence the function

$$\mathfrak{S}(x) = \theta(x) - 2\gamma_x(x) + \frac{2}{3}a_{xx}(x) \quad (58)$$

is invariant under projective transformations since both pencils are. Its variation under an arbitrary diffeomorphism gives us a cocycle which vanishes on projective transformations.

Using the equations of transformation for  $\gamma$ ,  $\theta$  and  $a$  (see appendix A) we come to the fact that under the diffeomorphism  $y = y(x)$ ,

$$\mathcal{S}|Dx|^2 \mapsto \mathcal{S}|Dx|^2 - \left( \frac{y_{xxx}}{y_x} - \frac{3}{2} \left( \frac{y_{xx}}{y_x} \right)^2 \right) a(x)|Dx|^2,$$

and we see that we come to the Schwarzian. The appearance of this ubiquitous object here may not be too surprising, we shall delay a more detailed study of this until later.

#### APPENDIX A. CONNECTIONS AND UPPER CONNECTIONS ON DENSITIES, AND BRANS-DICKE FUNCTIONS

In this Appendix we will give a brief recollection of the geometrical objects which appear in this article. For more details see [6].

A connection  $\nabla$  on the algebra of densities defines a covariant derivative of densities with respect to vector fields. It obeys the natural linearity properties and the Leibnitz rule:

- $\nabla_{\mathbf{X}}(\mathbf{s}_1 + \mathbf{s}_2) = \nabla_{\mathbf{X}}(\mathbf{s}_1) + \nabla_{\mathbf{X}}(\mathbf{s}_2)$ ,
- $\nabla_{f\mathbf{X}+g\mathbf{Y}}(\mathbf{s}) = f\nabla_{\mathbf{X}}(\mathbf{s}) + g\nabla_{\mathbf{Y}}(\mathbf{s})$ ,
- $\nabla_{\mathbf{X}}(\mathbf{s}_1\mathbf{s}_2) = \nabla_{\mathbf{X}}(\mathbf{s}_1)\mathbf{s}_2 + \mathbf{s}_1\nabla_{\mathbf{X}}(\mathbf{s}_2)$  (in particular  $\nabla_{\mathbf{X}}(f\mathbf{s}) = (\partial_{\mathbf{X}}f)\mathbf{s} + f\nabla_{\mathbf{X}}(\mathbf{s})$ ),

for arbitrary densities  $\mathbf{s}$ ,  $\mathbf{s}_1$  and  $\mathbf{s}_2$ , arbitrary vector fields  $\mathbf{X}$  and  $\mathbf{Y}$ , and arbitrary functions  $f$  and  $g$ . Here  $\partial_{\mathbf{X}}$  is the ordinary derivative of a function along a vector field.

Denote by  $\nabla_i$  the covariant derivative with respect to the vector field  $\partial_i = \partial/\partial x^i$ . For an arbitrary density  $\mathbf{s} = s(x)|Dx|^\lambda$  of weight  $\lambda$ ,

$$\nabla_i \mathbf{s} = (\partial_i s + \lambda \gamma_i s) |Dx|^\lambda, \quad \text{where } \gamma_i(x)|Dx| = \nabla_i(|Dx|).$$

Under a change of local coordinates  $x^i = x^i(x')$ , the symbol  $\gamma_a$  transforms in the following way:

$$\gamma_i = x_i^{i'} \left( \gamma_{i'} + \partial_{i'} \log \left| \det \frac{\partial x}{\partial x'} \right| \right) = x_i^{i'} \gamma_{i'} - x_{j'}^j x_{ij}^{j'}.$$

We use the shorthand notations for partial derivatives:  $x_i^{i'} = \partial x^{i'}/\partial x^i$  and  $x_{jk}^{i'} = \partial^2 x^{i'}/\partial x^j \partial x^k$ .

The connection also defines the divergence of a vector field

$$\operatorname{div}_{\nabla} \mathbf{X} = \partial_i X^i - \gamma_i X^i.$$

If the connection  $\nabla$  is induced by a volume form  $\rho$ ,  $\nabla = \nabla^\rho$  then  $\operatorname{div}_{\nabla} \mathbf{X} = \operatorname{div}_{\rho} \mathbf{X} = \rho^{-1} \partial_i (\rho X^i)$ .

Let  $S^{ij}$  be a contravariant tensor field. One can consider a *contravariant derivative* or an *upper connection*  ${}^S\nabla$  on densities associated with  $S$ . This notion can be defined by axioms similar to those for a usual connection. In particular on volume forms (densities of weight  $\lambda = 1$ ), we have

$${}^S\nabla^i \rho = {}^S\nabla^i (\rho(x)|Dx|) = (S^{ij} \partial_j \rho + \gamma^i \rho) |Dx|. \quad (59)$$

Given a contravariant tensor field  $S^{ij}$ , a connection  $\nabla$  (covariant derivative) induces an upper connection (contravariant derivative)  ${}^S\nabla$  by the rule  ${}^S\nabla^i = S^{ij} \nabla_j$ . If the tensor field  $S^{ij}$  is non-degenerate, the converse is also true. A non-degenerate contravariant tensor field  $S^{ij}(x)$  induces a one-to-one correspondence between upper connections and usual connections.

Under a change of coordinates the symbol  $\gamma^i$  for an upper connection (59) transforms as follows:

$$\gamma^{i'} = x_i^{i'} \left( \gamma^i + S^{ij} \partial_j \log \left| \det \frac{\partial x'}{\partial x} \right| \right). \quad (60)$$

It is worth noting that the difference of two connections on volume forms is a covector field, the difference of two upper connections on volume forms is a vector field. In other words the space of all

connections (upper connections) is an affine space associated with the linear space of the covector (vector) fields.

Consider two important examples of connections on volume forms.

**Example A.1.** An arbitrary volume form  $\rho$  defines a connection  $\nabla^\rho$  by the formula  $\Gamma_i = \Gamma_i^\rho = -\partial_i \log \rho(x)$ . If  $\mathbf{s} = s(x)|Dx|^\lambda$  is a density of weight  $\lambda$  then

$$\nabla_{\mathbf{X}}^\rho \mathbf{s} = \rho^\lambda \partial_{\mathbf{X}} \left( \rho^{-\lambda} \mathbf{s} \right) = X^i (\partial_i s + \lambda \Gamma_i s) |Dx|^\lambda.$$

This is a flat connection, i.e. its curvature vanishes:  $F_{ij} = \partial_i \gamma_j - \partial_j \gamma_i = 0$ .

**Example A.2.** Let  $\nabla^{TM}$  be an affine connection on a manifold  $M$  (i.e., a connection on the tangent bundle). It defines a connection on volume forms  $\nabla = -\text{Tr} \nabla^{TM}$  with  $\gamma_a = -\Gamma_{ab}^b$  where  $\Gamma_{bc}^a$  is the Christoffel symbol for  $\nabla^{TM}$ .

Finally we shall mention Brans-Dicke functions. If  ${}^S\nabla$  is an upper connection for the symmetric contravariant tensor field  $S^{ik}$  then an object  $\theta$  is a Brans-Dicke function corresponding to the upper connection  ${}^S\nabla$  if  $\theta - 2\gamma^i \Gamma_i + S^{ik} \Gamma_i \Gamma_k$  is a scalar function, where  $\Gamma_i$  is an arbitrary connection. (If one changes the connection  $\Gamma_i$  to another,  $\tilde{\Gamma}_i$ , then an expression  $\theta - 2\gamma^i \Gamma_i + S^{ik} \Gamma_i \Gamma_k$  changes by a scalar). It is easy to see that if the upper connection  $\gamma^i$  corresponds to a genuine connection  $\gamma_i$ ,  $\gamma^i = S^{ik} \gamma_k$ , then the function  $\gamma^i \gamma_i$  is a Brans-Dicke function.

Under changing of coordinates the Brans-Dicke function transforms as follows:

$$\theta' = \theta + 2\gamma^i \partial_i \log J + \partial_i \log J S^{ij} \partial_j \log J, \text{ where } J = \log \det \left( \frac{\partial x'}{\partial x} \right).$$

Brans-Dicke functions naturally arise when analysing second order operators. They play the role of second order connections. (See section 2.2, [6] and also equation (58).)

## APPENDIX B. PROOF OF LEMMA 1

Without loss of generality suppose that  $F$  is a function on operators which act on rapidly decreasing functions on  $d$ -dimensional affine space  $\mathbb{R}^d$  ( $d \geq 3$ ), and we take the volume form  $\rho = |Dx|$  in chosen Cartesian coordinates  $x^i$ .

A differential polynomial  $F(\Delta)$  which is invariant with respect to the algebra  $\text{sdiff}_\rho(\mathbb{R}^n)$  of divergenceless vector fields has to be invariant with respect to its subalgebra  $\text{saff}(m, \mathbb{R})$  of translations and divergenceless linear transformations. It is the subalgebra generated by the vector fields  $\partial_i$  and  $x^i \partial_j - \frac{1}{m} \delta_j^i x^k \partial_k$  ( $i, j, k = 1 \dots, d$ ) (translations and infinitesimal unimodular transformations). Due to classical results of invariant theory, the algebra of invariant tensors is generated by  $\delta_k^i$  and  $\varepsilon^{i_1 \dots i_d}$ . Since the dimension  $d$  is greater than 3 all invariant differential polynomials are linear combinations of the coefficients of operators and traces of their derivatives. E.g. a differential polynomial  $F$  on second order operators which is  $\text{saff}$ -equivariant has the following appearance:

$$F(S^{ik} \partial_i \partial_k + T^i \partial_i + R) = a_1 S^{ik} \partial_i \partial_k + a_2 \partial_i S^{ik} \partial_k + a_3 \partial_i \partial_k S^{ik} + b_1 T^i \partial_i + b_2 \partial_i T^i + cR, \quad (61)$$

where  $a_1, a_2, a_3, b_1, b_2, c$  are arbitrary constants<sup>4</sup>.

Now let  $F$  be a linear  $\text{sdiff}_\rho$ -equivariant differential polynomial, on the space  $\mathcal{D}^{(n)}(\mathbb{R}^m)$ :

$$\text{ad}_{\mathbf{X}} F(\Delta) = F(\text{ad}_{\mathbf{X}} \Delta),$$

for an arbitrary divergenceless vector field  $\mathbf{X}$ , ( $\text{div}_\rho \mathbf{X} = \partial_i X^i = 0$ .) We prove that it implies that

$$F(\Delta) = a\Delta + b\Delta^+ + c\Delta(1) + d\Delta^+(1). \quad (62)$$

<sup>4</sup>In the case of dimension  $d = 2$  the invariant tensor  $\varepsilon^{ik}$  allows to consider maps such as  $F(A^i \partial_i + T) = a\varepsilon^{im} \partial_i A^m \partial_m \partial_n + b\varepsilon^{mn} \partial_m R \partial_n$ .

We use induction on the order of operators. First we check by straightforward calculations that this is true for  $n = 1, 2$ . Indeed if  $n = 2$  then  $F$  has the appearance (61), since it is necessarily saff  $(m, \mathbb{R})$ -equivariant. One can then proceed by direct calculations. For this it is useful to use that

$$\text{ad}_{\mathbf{X}}(T^i \partial_i) = (\mathcal{L}_K T^i) \partial_i, \quad \text{and } \text{ad}_{\mathbf{X}}(S^{ij} \partial_i \partial_j) = (\mathcal{L}_{\mathbf{X}} S^{ij}) \partial_i \partial_j - S^{ij} \partial_i \partial_j K^r \partial_r$$

for a divergence-less vector field  $\mathbf{X}$ ,  $(\partial_i X^i = 0)$ .

Comparing the expressions for  $F(\text{ad}_{\mathbf{X}} \Delta)$  and  $\text{ad}_{\mathbf{X}} F(\Delta)$ , where  $F$  is defined by (61), we come to the requirement that  $b_1 = a_1 - a_2$  and  $b_2 = -a_3$ . This implies equation (62).

The statement that we want to prove for arbitrary  $n$  immediately follows from the following observation.

**Lemma 2.** *Let  $F$  be a linear sdiff  $\rho$ -equivariant map on  $\mathcal{D}^{(n)}(\mathbb{R}^m)$  which depends only on the principal symbol of operator  $\Delta$ . Then  $F$  vanishes:  $F = 0$  if  $n \geq 3$ .*

The result follows from above as given  $F$ , a linear sdiff  $\rho$ -equivariant map on  $\mathcal{D}^{(n)}(\mathbb{R}^m)$  for  $n \geq 3$ , we have that, by the inductive hypothesis, the restriction of  $F$  to the subspace  $\mathcal{D}^{(n-1)}(\mathbb{R}^m)$  obeys the condition (62), i.e.  $F = a\Delta + b\Delta^+ c\Delta(1) + d\Delta^+(1)$  on  $\mathcal{D}^{(n-1)}(\mathbb{R}^m)$ . Hence the map  $F' = F - a\Delta + b\Delta^+ c\Delta(1) + d\Delta^+(1)$  on  $\mathcal{D}^{(n)}(\mathbb{R}^m)$  depends only on the principal symbol of the operator  $\mathcal{D}$ . Due to the lemma 2,  $F' = 0$ , i.e.  $F$  has the appearance (62).

It remains to prove lemma (2). Suppose  $F(\Delta)$  on  $\mathcal{D}^{(n)}$  depends only on principal symbol of  $\Delta$ :

$$F(S^{i_1 \dots i_n} \partial_{i_1} \dots \partial_{i_n} + \dots) = a_0 S^{i_1 \dots i_n} \partial_{i_1} \dots \partial_{i_n} + a_1 \partial_{i_1} S^{i_1 i_2 \dots i_n} \partial_{i_2} \dots \partial_{i_n} + a_2 \partial_{i_1} \partial_{i_2} S^{i_1 i_2 i_3 \dots i_n} \partial_{i_3} \dots \partial_{i_n} + \dots$$

It suffices to prove that  $a_0 = 0$ . Namely suppose that  $a_0 = a_1 = a_{k-1} = 0$  ( $k \geq 1$ ), then consider the map  $S^{i_1 \dots i_n} \rightarrow a_k \partial_{i_1} \dots \partial_{i_k} S^{i_1 \dots i_k i_{k+1} \dots i_n}$ . If  $a_1 = \dots = a_{k-1} = 0$  then this is a map from the principal symbol of  $\Delta$  to the principal symbol of the operator  $F(\Delta)$ . Hence this is sdiff  $\rho$ -equivariant map. On the other hand this map is not sdiff  $\rho$ -equivariant if  $a_k \neq 0$  (if  $n \geq 2$ ). Hence  $a_k = 0$  also. Thus we have shown that from the fact that  $a_0 = 0$  it follows that  $F = 0$ .

One can prove that the condition  $a_0 = 0$  is necessary condition by straightforward calculations, comparing the expressions for  $\text{ad}_{\mathbf{X}} F(\Delta)$  and  $F(\text{ad}_{\mathbf{X}} \Delta)$ . These calculations can be essentially facilitated if we consider the map  $F$  on operators such that  $\partial_{i_1} S^{i_1 i_2 \dots i_n} \partial_{i_2} \dots \partial_{i_n}$  identically vanishes in a vicinity of given point. We have that in this case  $F(\text{ad}_{\mathbf{X}} \Delta) = F((\mathcal{L}_{\mathbf{X}} S)^{i_1 \dots i_n} \partial_{i_1} \dots \partial_{i_n}) =$

$$a_0 (\mathcal{L}_{\mathbf{X}} S)^{i_1 \dots i_n} \partial_{i_1} \dots \partial_{i_n} + a_1 \partial_p (\mathcal{L}_{\mathbf{X}} S)^{p i_2 \dots i_n} \partial_{i_2} \dots \partial_{i_n} + a_2 \partial_p \partial_q (\mathcal{L}_{\mathbf{X}} S)^{p q i_3 \dots i_n} \partial_{i_3} \dots \partial_{i_n} + \dots =$$

$$a_0 \mathcal{T}_0 - a_1 (n-1) \mathcal{T}_1 - a_2 ((n-2) \mathcal{T}_2 - \mathcal{T}_3) + \dots$$

and

$$\text{ad}_{\mathbf{X}} F(\Delta) = \text{ad}_{\mathbf{X}} (a_0 S^{i_1 \dots i_n} \partial_{i_1} \dots \partial_{i_n}) = a_0 \left( \mathcal{T}_0 - \frac{n(n-1)}{2} \mathcal{T}_1 - \frac{n(n-1)(n-2)}{6} \mathcal{T}_2 + \dots \right)$$

where we denote  $\mathcal{T}_0 = (\mathcal{L}_{\mathbf{X}} S)^{i_1 \dots i_n} \partial_{i_1} \dots \partial_{i_n}$ ,  $\mathcal{T}_1 = S^{p q i_3 \dots i_n} \partial_p \partial_q K^{i_2} \partial_{i_2} \dots \partial_{i_n}$ ,  $\mathcal{T}_2 = S^{p q s i_4 \dots i_n} \partial_p \partial_q \partial_s K^{i_3} \partial_{i_3} \dots \partial_{i_n}$  and  $\mathcal{T}_3 = \partial_p S^{q s i_3 \dots i_n} \partial_q \partial_s K^p \partial_{i_3} \dots \partial_{i_n}$ . Comparing the terms proportional to  $\mathcal{T}_2$  and  $\mathcal{T}_3$  we see that on one hand  $a_2 = \frac{n(n-1)}{6} a_0$  and on the other hand  $a_2 = 0$ . Hence  $a_0 = 0$ .

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