KALUZA-KLEIN THEORY REVISITED: PROJECTIVE STRUCTURES AND DIFFERENTIAL OPERATORS ON ALGEBRA OF DENSITIES.

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ABSTRACT. We consider differential operators acting on densities of arbitrary weights on manifold M identifying pencils of such operators with operators on algebra of densities of all weights. This algebra can be identified with the special subalgebra of functions on extended manifold $\widehat{M}$. On one hand there is a canonical lift of projective structures on M to affine structures on extended manifold $\widehat{M}$. On the other hand the restriction of algebra of all functions on extended manifold to this special subalgebra of functions implies the canonical scalar product. This leads in particular to classification of second order operators with use of Kaluza-Klein-like mechanisms.

1. Algebra of densities

In mathematical physics it is very useful to consider differential operators acting on densities of various weights on a manifold M (see [7] and citations there). We say that $\boldsymbol{s} = s(x)|Dx|^\lambda$ is a density of weight λ on M if under change of local coordinates

$$\boldsymbol{s} = s(x)|Dx|^\lambda = s\left(x\left(x'\right)\right)\left|\det\left(\frac{\partial x}{\partial x'}\right)\right|^\lambda |Dx'|^\lambda\,,$$

(λ is an arbitrary real number). We denote by $\mathcal{F}_\lambda(M)$ the space of densities of weight λ on manifold M. (The space of functions on M is $\mathcal{F}_0(M)$, densities of weight $\lambda = 0$.)

Densities can be multiplied. If $\boldsymbol{s}_1 = s_1(x)|Dx|^{\lambda_1}$ and $\boldsymbol{s}_2 = s_2(x)|Dx|^{\lambda_2}$ are densities of weights λ_1, λ_2 respectively then $\boldsymbol{s} = \boldsymbol{s}_1 \cdot \boldsymbol{s}_2 = s_1(x)s_2(x)|Dx|^{\lambda_1+\lambda_2}$ is a density of weight $\lambda_1 + \lambda_2$. We come to the algebra $\mathcal{F}(M) = \oplus_\lambda \mathcal{F}_\lambda(M)$ of finite linear combinations of densities of arbitrary weights. Use a formal variable t instead volume form $|Dx|$. Thus an arbitrary density $\boldsymbol{s} = s_1(x)|Dx|^{\lambda_1} + \cdots + s_k|Dx|^{\lambda_k}$ can be written as a function on x, t which is *quasipolynomial* on t, $\boldsymbol{s}(x,t) = s_1(x)t^{\lambda_1} + \cdots + s_k(x)t^{\lambda_k}$. An arbitrary density $\boldsymbol{s} \in \mathcal{F}(M)$ can be identified with function $\sum s_r(x)t^{\lambda_r}$ on the extended manifold $\widehat{M}$, which is quasipolynomial on 'vertical' variable t. There is a natural fibre bundle structure $\widehat{M} \to M$. Extended manifold $\widehat{M}$ is the frame bundle of the determinant bundle of M, (x^i, t) are local coordinates on $\widehat{M}$. Changing of local coordinates is:

$$(x^{i'}, t')\colon \quad x^{i'} = x^{i'}(x^i), \quad t' = t'(x^i, t) = \det\left(\frac{\partial x^{i'}}{\partial x^i}\right) t\,.$$

The fibre bundle $\widehat{M} \to M$ can be used for studying projective geometry on M since there is a canonical construction which assigns to an arbitrary projective connection on manifold

2000 *Mathematics Subject Classification.* 15A15, 58A50, 81R99.

Key words and phrases. differential operator, algebra of densities, pencil of operators, self-adjoint operators, equivariant maps on operators.

M an usual affine connection on $\widehat{M}$ (see the last section). This affine connection on $\widehat{M}$ can be used for describing the 'projective geometry' on the base manifold M. Such investigation can be traced to H.Weyl, Weblen and Thomas. On the other hand we will come to additional geometrical structures on fibre bundle $\widehat{M} \to M$ if instead algebra of all smooth functions on extended manifold $\widehat{M}$ we consider only the subalgebra of functions on $\widehat{M}$, which are quasipolynomial on vertical variable t, i.e. algebra $\mathcal{F}(M)$ of densities on M. This algebra can be equipped with the canonical scalar product: If $\boldsymbol{s}_1 = s_1(x)|Dx|^{\lambda_1}$ and $\boldsymbol{s}_2 = s_2(x)|Dx|^{\lambda_2}$ are two densities with a compact support then

$$\langle \boldsymbol{s}_1, \boldsymbol{s}_2 \rangle = \begin{cases} \int_M s_1(x)s_2(x)|Dx|\,, & \text{if} \quad \lambda_1+\lambda_2 = 1\,, \\ \\ 0 \qquad \text{if} & \lambda_1+\lambda_2 \neq 1\,. \end{cases} \tag{1}$$

This construction turns out to be very important tool to study geometry of differential operators on M [4], [5]. We give short exposition of these results. Here we come to these results and formulate new ones based on alternative approach of Kaluza-Klein-like reduction.

2. Differential operators on algebra of densities

Consider linear operator $\widehat{w}$ such that $\widehat{w}(\boldsymbol{s}) = \lambda \boldsymbol{s}$ in the case if $\boldsymbol{s}$ is a density of weight λ, ($\boldsymbol{s} \in \mathcal{F}_\lambda(M)$). If $\boldsymbol{s}_1$ is a density of weight λ_1 and $\boldsymbol{s}_2$ is a density of weight λ_2 then

$$\widehat{w}(\boldsymbol{s}_1 \cdot \boldsymbol{s}_2) = (\lambda_1+\lambda_2)\boldsymbol{s}_1 \cdot \boldsymbol{s}_2 = (\widehat{w}\boldsymbol{s}_1) \cdot \boldsymbol{s}_2 + \boldsymbol{s}_1 \cdot (\widehat{w}\boldsymbol{s}_2)\,.$$

Leibnitz rule is obeyed, $\widehat{w}$ is first order differential operator on the algebra of densities. In local coordinates (x^i, t) on $\widehat{M}$, $\widehat{w} = t\frac{\partial}{\partial t}$. A differential operator $\widehat{\Delta}$ on algebra of densities has appearance $\widehat{\Delta} = \widehat{\Delta}\left(x, \frac{\partial}{\partial x}, t, \widehat{w}\right)$ in local coordinates. An arbitrary operator $\widehat{\Delta}$ on algebra of densities defines the pencil of operators:

$$\widehat{\Delta} \mapsto \{\Delta_\lambda\}\colon \Delta_\lambda = \widehat{\Delta}\big|_{\widehat{w}=\lambda}\,.$$

E.g. the operator $\widehat{\Delta} = A(\widehat{w})S^{ik}\partial_i\partial_k + B(\widehat{w})T^i\partial_i + C(\widehat{w})R$ on algebra $\mathcal{F}(M)$ defines the pencil of operators $\{\Delta_\lambda\}$: $\Delta_\lambda = A(\lambda)S^{ik}\partial_i\partial_k + B(\lambda)T^i\partial_i + C(\lambda)R$. Here A, B, C are polynomials on $\widehat{w}$. If for example $A = 1+\widehat{w}$, $B = \widehat{w}^2$ and $C = 1$, then $\widehat{\Delta}$ is third order operator on the algebra of densities, which defines the pencil of second order operators. Operators on algebra of densities can be identified with operator pencils which depend polynomially on pencil parameter λ.

Remark 1. Here we consider only operators which do not change the weight of densities: $\widehat{\Delta} = \widehat{\Delta}(x, \frac{\partial}{\partial x}, \widehat{w})$, i.e. for corresponding pencil $\{\Delta_\lambda\}$, $\Delta_\lambda\colon \mathcal{F}_\lambda(M) \to \mathcal{F}_\lambda(M)$.

Canonical scalar product (1) defines adjointness of linear operators. Linear operator $\widehat{\Delta}$ acting on the algebra of densities has an adjoint $\widehat{\Delta}^*$: $\langle \widehat{\Delta}\boldsymbol{s}_1, \boldsymbol{s}_2 \rangle = \langle \boldsymbol{s}_1, \widehat{\Delta}^*\boldsymbol{s}_2 \rangle$. One can see that $(x^i)^* = x^i$, $\partial_i^* = -\partial_i$ and $\widehat{w}^* = 1 - \widehat{w}$.

To consider self-adjoint and anti-self-adjoint operators on extended manifold $\widehat{M}$ is very illuminating for studying geometry of operators on base manifold M. (See for details [4], [5],[1] and [6].)

3. First order operators and Kaluza-Klein mechanism

Consider an arbitrary first order operator $\widehat{K}$ such that it does not change the weight of densities (see Remark 1) and obeys normalisation condition $\widehat{K}(1) = 0$. In local coordinates it has the following appearance

$$\widehat{K} = K^i(x)\partial_i + K^0(x)\widehat{w}\,.$$

One can see that its adjoint is equal to $\widehat{K}^* = -K^i(x)\partial_i - \partial_i K^i(x) + K^0(x)(1-\widehat{w})$. $\widehat{K}$ is a vector field on extended manifold $\widehat{M}$. One can define divergence of this vector field:

$$\operatorname{div}\widehat{K} = -(\mathbf{K}+\mathbf{K}^*) = \partial_i K^i(x) - K^0(x)\,. \tag{2}$$

Notice that the divergence is defined in spite of the absence of well-defined volume form on $\widehat{M}$ (see for details [4] and [6]).

We see that vector field $\widehat{K}$ is divergenceless iff it is anti-self-adjoint:

$$\widehat{K} = -\widehat{K}^* \Leftrightarrow \operatorname{div}\widehat{K} = 0 \Leftrightarrow \widehat{K} = K^i(x)\partial_i + \widehat{w}\partial_i K^i(x)\,.$$

One can see that divergenceless vector field $\widehat{K}$ is a Lie derivative of densities along its projection, vector field $\mathbf{K}$ on M. An arbitrary vector field $\mathbf{X} = X^i(x)\partial_i$ can be lifted to anti-self-adjoint (i.e. divergenceless) vector field on extended manifold $\widehat{M}$, which is nothing but Lie derivative of densities: $\mathbf{X} \mapsto \widehat{\mathfrak{L}}_{\mathbf{X}}$ such that for arbitrary $\boldsymbol{s} = s(x)|Dx|^\lambda$,

$$\widehat{\mathfrak{L}}_{\mathbf{X}}(\boldsymbol{s}) = \widehat{X}(\boldsymbol{s}) = \big(X^i(x)\partial_i + \widehat{w}\partial_i X^i(x)\big)\, s(x)|Dx|^\lambda = \big(X^i\partial_i s(x) + \lambda\partial_i X^i s(x)\big)\,|Dx|^\lambda\,.$$

It is useful to consider a connection on a bundle $\widehat{M} \to M$. It assigns to every vector field $\mathbf{X} = X^i(x)\partial_i$ on M its lifting, the horizontal vector field $\widehat{X}_{\rm hor} = X^i(x)\partial_i + \gamma_i(x)X^i(x)\widehat{w}$ on $\widehat{M}$. Connection defines derivation $\nabla_{\mathbf{X}}$ on algebra of densities: for $\boldsymbol{s} = s(x)|Dx|^\lambda$

$$\nabla_{\mathbf{X}}(\boldsymbol{s}) = \widehat{X}_{\rm hor}(\boldsymbol{s}) = \big(X^i(x)\partial_i s(x) + \lambda\gamma_i(x)X^i(x)s(x)\big)\,|Dx|^\lambda\,. \tag{3}$$

Under changing of local coordinates $x^i \to x^{i'} = x^{i'}(x^i)$ components γ_i of connection are transforming in the following way:

$$\gamma_{i'} = \frac{\partial x^i}{\partial x^{i'}}\left(\gamma_i + \frac{\partial}{\partial x^i}\log\left(\det\frac{\partial x^{j'}}{\partial x^j}\right)\right)\,, \quad \left(\gamma_i(x)|Dx| = \nabla_i|Dx|\right)\,.$$

Connection $\gamma_i(x)$ defines divergence $\operatorname{div}_\gamma$ of vector fields on M, which is equal to divergence (2) of horizontal lifting of this vector field: $\operatorname{div}_\gamma\mathbf{X} = \operatorname{div}\widehat{X}_{hor} = \partial_i X^i(x) - \gamma_i(x)X^i(x)$.

Remark 2. Let $\mathbf{X} = X^i\partial_i$ be a projection on M of a vector field $\widehat{X} = X^i\partial_i + \widehat{w}X^0$ on M, and $\widehat{X}_{\rm hor}$ be a horizontal lifting of $\mathbf{X}$. Then $\widehat{X} - \widehat{X}_{\rm hor} = \widehat{w}(X^0 - \gamma_i X^i)$ is a vertical vector field and $X^0 - \gamma_i X^i$ is a scalar field.

Remark 3. A volume form $\boldsymbol{\rho} = \rho(x)|Dx|$ on M naturally defines a connection $\gamma_i = -\partial_i \log\rho(x)$. A Riemannian metric $G = g_{ik}dx^i dx^k$ on M naturally defines a volume form $\boldsymbol{\rho} = \sqrt{\det g}|Dx|$. The corresponding connection $\gamma_i = -\Gamma^k_{ik}$, where Γ^i_{km} are Cristoffel symbols of Levi-Civita connection of the metric. In this case $\operatorname{div}_\gamma$ is a standard divergence operator (with respect to a volume form).

4. Second order operators and Kaluza-Klein reduction

Let $\widehat{\Delta}$ be an arbitrary second order operator on algebra of densities $\mathcal{F}(M)$:

$$\widehat{\Delta} = \underbrace{S^{ik}(x)\partial_i\partial_k + 2\widehat{w}B^i(x)\partial_i + \widehat{w}^2 C(x)}_{\text{second order derivatives}} + \underbrace{D^i(x)\partial_i + \hat{w}E(x)}_{\text{first order derivatives}} + F(x)\,. \tag{4}$$

(As always we consider only operators which do not change weight of densities (see remark 1).)

Principal symbol of this operator is

$$\widehat{S} = \begin{pmatrix} S^{ik} & B^i \\ B^k & C \end{pmatrix}, \qquad (\text{in coordinates } x^i, x^0 = \log t)\,.$$

$\widehat{S}$ is a contravariant symmetric tensor field on the extended manifold $\widehat{M}$. 'Space components' S^{ik} of the tensor field $\widehat{S}$ are components of symmetric contravariant tensor field on M. Operator $\widehat{\Delta}$ defines a pencil of second order operators $\{\Delta_\lambda\}$, $\Delta_\lambda = S^{ik}\partial_i\partial_k + \ldots$, and all these operators have the same principal symbol S^{ik}.

Put normalisation condition $F = \widehat{\Delta}(1) = 0$ and consider the operator which is adjoint to operator (4):

$$\widehat{\Delta}^* = \partial_k\partial_i\left(S^{ik}\ldots\right) - \widehat{w}^*\partial_i\left(2B^i + (\ldots)\right) + (\widehat{w}^*)^2\left(C\ldots\right) - \partial_i\left(D^i\ldots\right) + \widehat{w}^*E\,,\ (\widehat{w}^* = 1 - \widehat{w})\,.$$

The condition that operator $\widehat{\Delta}$ is self-adjoint, $\widehat{\Delta}^* = \widehat{\Delta}$ implies that

$$\widehat{\Delta} = S^{ik}\partial_i\partial_k + \partial_k S^{ki}\partial_i + (2\widehat{w} - 1)\,B^i\partial_i + \hat{w}\partial_i B^i + \widehat{w}\,(\widehat{w} - 1)\,C\,. \tag{5}$$

Thus self-adjoint second order operator on algebra of densities, which obeys normalisation condition $\widehat{\Delta}(1) = 0$ is uniquely defined by its symbol.

The geometry of operator (5) was studied in detail in articles [4], [5] and [1]. Here we present and analyze these results, using Kaluza-Klein-like mechanism.

Kaluza-Klein mechanism defines a connection (gauge field) and Riemannian metric on a base manifold through Riemannian metric on a total space of fibre bundle $\widehat{M} \to M$. Connection, i.e. the distribution of horizontal hyperplanes (subspaces which are transversal to the fibres) is defined by the condition that these hyperplanes are orthogonal to the fibres with respect to Riemannian metric in the bundle space.

One can slightly alter this mechanism. Contravariant tensor field $\widehat{S}$, principal symbol of operator (5) maps 1-forms (covectors) to vectors on $\widehat{M}$. Consider the following Kaluza-Klein-like mechanism: take an arbitrary 1-form Ω on $\widehat{M}$ such that $\Omega(\widehat{w}) \neq 0$, i.e. Ω is proportional to form $dx^0 + \ldots$ ($x^0 = \log t$), and the following condition is obeyed: vector field $\widehat{S}\Omega$ is proportional to vertical vector field $\widehat{w}$ ($\widehat{w} = t\frac{\partial}{\partial t} = \frac{\partial}{\partial x^0}$) . This means that for 1-form $\Omega = a(x)(dx^0 - \gamma_k(x)dx^k)$ the following condition holds:

$$\begin{pmatrix} S^{ik} & B^i \\ B^k & C \end{pmatrix}\begin{pmatrix} -\gamma_k \\ 1 \end{pmatrix} \text{ is proportional to vector } \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{6}$$

This condition canonically defines distribution of horizontal hyperplanes in $\widehat{M}$, which are sets of vectors which annihilate the form Ω. (One can take $\widehat{S}\Omega = 0$ in the case if $\widehat{S}$ is degenerate.) Every vector field $\mathbf{X} = X^i(x)(x)\partial_i$ on the base M can be lifted to horizontal vector field which annihilates the connection form Ω (see also equation (3)). This construction works if condition (6) is obeyed, i.e. in the case if the equation

$$S^{ik}(x)\gamma_k(x) = B^i(x) \tag{7}$$

has a solution. In this case second order operator (4) via its principal symbol $\widehat{S}$ defines a connection γ_k. In the case if S^{ik} is non-degenerate then an equation (7) has unique solution. In this case operator defines uniquely canonical connection and Riemannian metric on the base.

The field $B^i(x) = S^{ik}(x)\gamma_k(x)$ can be considered as an *upper connection*. It follows from (6) that in this case $C - B^i\gamma_i$ is a scalar. and C is related with *Brans-Dicke function.*

In general case if the condition (7) is not obeyed then more detailed analysis shows that $B^i - S^{ik}\Gamma_k$ is a vector field and $C - 2B^i\Gamma_i + S^{ik}\Gamma_i\Gamma_k$ is a scalar, where Γ_i is an arbitrary connection.

The importance of operator (5) is defined by the following uniqueness Theorem:

Theorem 1. *Let Δ be second order operator acting on densities of weight λ_0, where $\lambda_0 \neq 0, 1, 1/2$.*

Then there exists a unique self-adjoint operator $\widehat{\Delta}$ ($\widehat{\Delta}^* = \widehat{\Delta}$) which obeys the following conditions

- $\widehat{\Delta}\big|_{\hat{w}=\lambda_0} = \Delta$,
- *normalisation condition* $\widehat{\Delta}(1) = 0$.

In other words there exists unique self-adjoint normalised pencil of second order operators which passes through a given operator.

This Theorem was formulated and proved in [4] (see also [5]).

Example 4.1. Consider second order operator $\Delta_{\lambda_0} = \mathfrak{L}_{\mathbf{X}} \circ \mathfrak{L}_{\mathbf{Y}}$, where $\mathbf{X}, \mathbf{Y}$ are arbitrary vector fields, $\mathfrak{L}_{\mathbf{X}}, \mathfrak{L}_{\mathbf{Y}}$ are Lie derivatives of densities of weight λ_0 along vector fields $\mathbf{X}, \mathbf{Y}$ respectively. Construct the following operator on algebra of densities:

$$\widehat{\Delta} = \frac{1}{2}\left(\widehat{\mathfrak{L}}_{\mathbf{X}}\widehat{\mathfrak{L}}_{\mathbf{Y}} + \widehat{\mathfrak{L}}_{\mathbf{Y}}\widehat{\mathfrak{L}}_{\mathbf{X}}\right) + \frac{1}{2}\left(\frac{2\widehat{w} - 1}{2\lambda_0 - 1}\right)\left(\widehat{\mathfrak{L}}_{\mathbf{X}}\widehat{\mathfrak{L}}_{\mathbf{Y}} - \widehat{\mathfrak{L}}_{\mathbf{Y}}\widehat{\mathfrak{L}}_{\mathbf{X}}\right).$$

This operator is obviously self-adjoint operator and it passes through operator Δ_{λ_0}. One can see that it is equal to

$$\widehat{\Delta} = \widehat{\mathfrak{L}}_{\mathbf{X}}\widehat{\mathfrak{L}}_{\mathbf{Y}} + \left(\frac{\widehat{w} - \lambda_0}{2\lambda_0 - 1}\right)\widehat{\mathfrak{L}}_{[\mathbf{X},\mathbf{Y}]}.$$

Let $\Delta_{\lambda_0} = A^{ij}\partial_i\partial_j + A^i\partial_i + A$ be an arbitrary second order operator acting on space of densities of a given weight λ_0, ($\lambda_0 \neq 0, 1/2, 1$). Then we see that the self-adjoint operator (5) passes via the operator Δ_{λ_0} if for upper connection B^i and Brans-Dicke function C

$$B^i = \frac{A^i - \partial_k A^{ki}}{2\lambda_0 - 1}, C = \frac{A}{\lambda_0(\lambda_0 - 1)} - \frac{\partial_i A^i - \partial_i\partial_k A^{ki}}{(\lambda_0 - 1)(2\lambda_0 - 1)}.$$

The upper connection B^i is induced by a connection γ_i ($B^i = A^{ik}\gamma_k$) iff equation (7) has a solution (for $S^{ik} = A^{ik}$). The condition (7) defines this special property of second order operators on densities. It is interesting to analyze its geometrical meaning.

5. Thomas bundle and projective geometry

The canonical constructions studied in the previous sections were successfully performed since we consider not the algebra of all functions on extended manifold $\widehat{M}$, but only functions which are quasipolynomial on vertical variable t, since scalar product (1) is not well-defined on algebra of all (smooth) functions on x, t. Nevertheless in general case for fibre bundle $\widehat{M} \to M$ there exists the remarkable construction which assigns to projective class of connections on M the affine connection on $\widehat{M}$. This construction is due to T.Y.Thomas [8]. (See also [2] and [3]). The bundle $\widehat{M} \to M$ sometimes is called Thomas bundle. Now we sketch this construction.

We say that two symmetric affine connections ∇ and $\tilde{\nabla}$ on manifold M belong to the same projective class $[\nabla] = [\tilde{\nabla}]$ if

$$\tilde{\nabla} - \nabla = \tilde{\Gamma}^i_{km} - \Gamma^i_{km} = t_k \delta^i_m + t_m \delta^i_k \,, \qquad (t_i \text{ is covector})\,,$$

where $\tilde{\Gamma}^i_{km}$ and Γ^i_{km} are Christoffel symbols of connections $\tilde{\nabla}$ and ∇ respectively. Equivalence class of symmetric connections is projective connection. (Projective connection in particular defines non-parametrised geodesics: two symmetric affine connection belong to the same class iff they have the same non-parameterised geodesics.)

For affine connection ∇ on n-dimensional manifold M with Christoffel symbols Γ^i_{km} one can consider symbols

$$\Pi^i_{km}(\nabla) = \Pi^i_{km} = \Gamma^i_{km} + \frac{1}{n+1}\left(\gamma_k \delta^i_m + \gamma_m \delta^i_k\right)\,, \tag{8}$$

where $\gamma_i = -\Gamma^k_{ik}$ define connection on densities on M (see also remark 3). Two symmetric connections $\nabla, \tilde{\nabla}$ belong to the same projective class iff $\Pi^i_{km}(\nabla) = \Pi^i_{km}(\tilde{\nabla})$.

Let $[\nabla]$ be a projective class of symmetric connections on n-dimensional manifold M. Then Thomas construction assigns to this projective class $[\nabla]$ the symmetric affine connection $\widehat{\nabla}$ on the extended manifold $\widehat{M}$. with following Christoffel symbols

$$\widehat{\Gamma}^i_{km} = \Pi^i_{km},\, \widehat{\Gamma}^0_{km} = \frac{1}{n+1}\left(\partial_r \Pi^r_{km} - \Pi^r_{sk}\Pi^s_{rm}\right),\, \widehat{\Gamma}^i_{k0} = -\frac{\delta^i_k}{n+1},\, \widehat{\Gamma}^i_{00} = \widehat{\Gamma}^0_{i0} = 0,\, \widehat{\Gamma}^0_{00} = -\frac{1}{n+1}\,.$$

Here Π^i_{km} are symbols (8) corresponding to Christoffel symbols of a connection in the class $[\nabla]$. (We use local coordinates $(x^i, x^0) = (x^i, \log t)$) in the extended space.)

Acknowledgments This article is based on my talk at the conference "The Modern Physics of Compact Stars and Relativistic Gravity" (Yerevan, Septemebr 2013). I am grateful to organisers of this conference for financial support, and also to Engineering and Physical Sciences Research Council (EPSRC) for financial support via the Mathematics Platform Grant (MAPLE) EP/IO1912X/1. I am grateful to T.Voronov for fruitful discussions.

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