On odd Laplace operators *

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Abstract

We consider odd Laplace operators acting on densities of various weights on an odd Poisson (= Schouten) manifold M. We prove that the case of densities of weight $1/2$ (half-densities) is distinguished by the existence of a unique odd Laplace operator depending only on a point of an "orbit space" of volume forms. This includes earlier results for the odd symplectic case, where there is a canonical odd Laplacian on half-densities. The space of volume forms on M is partitioned into orbits by the action of a natural groupoid whose arrows correspond to the solutions of the quantum Batalin–Vilkovisky equations. We compare this situation with that of Riemannian and even Poisson manifolds. In particular, we show that the square of an odd Laplace operator is a Poisson vector field defining an analog of Weinstein's "modular class".

Contents

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Introduction

In this paper we study odd Laplace operators acting on densities of various weights on odd Poisson manifolds. This happens to be a very rich geometrical topic, surprisingly linking even and odd Poisson geometry with Riemannian geometry. The main result is the construction of odd Laplacians acting on half-densities (= densities of weight 1/2) in the context of Poisson geometry and a detailed study of their properties.

Earlier in [6, 7] it was shown that a canonical odd Laplacian on half-densities exists on odd symplectic manifolds. In the current paper we consider Laplacians on half-densities on arbitrary odd Poisson manifolds. This change of viewpoint has led us to a completely new picture.

Namely, rather than study a single Laplace operator, we consider all operators depending on arbitrary volume elements on the supermanifold. Laplace operators acting on half-densities are characterized by a commutator identity of the form $[\Delta, f] = \mathcal{L}_f$, where $\mathcal{L}_f$ is the Lie derivative along the Hamiltonian vector field with Hamiltonian f. Starting from this relation, we arrive at a natural groupoid whose orbits parametrize the set of odd Laplace operators: half-densities are distinguished because for them the Laplacian depends not on a volume element but only on its orbit. This groupoid, which we call the "master groupoid", appears also in other instances. It seems that it plays a very important role in odd Poisson geometry. Another interesting new object is a remarkable analog of Weinstein's "modular class", which comes from the square of odd Laplace operators.

Besides naturally expected links with even Poisson geometry, we have found far-reaching surprising links with Riemannian geometry. We found it worthy to point out at a simple analogy between the Batalin–Vilkovisky formalism in quantum field theory and the usual quantum mechanics, in particular, between the exponential of the "quantum master action" and the half-density wave function.

Remark. The Batalin–Vilkovisky formalism appeared around 1981 as a very general method of quantization of systems with gauge freedom [1, 2, 3]. The central role in this method is played by a second order odd differential operator somewhat similar to the divergence of multivector fields. From the viewpoint of supermanifolds it is better to interpret it as an analog of a Laplacian. Geometry of the Batalin–Vilkovisky (BV) formalism has been investigated in many works, in particular [9] and [4, 5]. Algebraic structures related with the BV formalism gradually became very fashionable. Invariant geometric constructions for the BV operator have been studied in [4], [9] in the symplectic case and recently in [8] in the Poisson case. In these works the operators considered act on functions. A new approach based on half-densities was suggested in [6, 7] and is developed here.

1 Some facts from odd Poisson geometry

1.1 Poisson brackets and Hamiltonian vector fields

Let us briefly recall some well known formulae, for reference purposes[1]. Given a supermanifold M, an arbitrary *odd Poisson* (or *Schouten*) *structure* on M is specified by an odd function on T^*M quadratic in momenta:

$$\mathfrak{S} = \frac{1}{2}\,\mathfrak{S}^{ab}(x)p_b p_a \tag{1.1}$$

("odd quadratic Hamiltonian"). Explicitly:

$$\{f,g\}_{\mathfrak{S}} := (f,(\mathfrak{S},g)) = ((f,\mathfrak{S}),g) = -(-1)^{\tilde{f}(\tilde{a}+1)}\mathfrak{S}^{ab}\frac{\partial f}{\partial x^b}\frac{\partial g}{\partial x^a}, \tag{1.2}$$

where we denote by $(\ ,\ )$ the canonical even Poisson bracket on T^*M. Here $\{f,g\}_{\mathfrak{S}}$ stands for the odd bracket on M specified by $\mathfrak{S}$. The Jacobi identity for $\{f,g\}_{\mathfrak{S}}$ is equivalent to the vanishing of the canonical Poisson bracket $(\mathfrak{S},\mathfrak{S}) = 0$ on T^*M.

In the sequel we leave $\mathfrak{S}$ fixed and drop the reference to $\mathfrak{S}$ from the notation for the bracket. The Hamiltonian vector field on M corresponding to a function f is defined as

$$X_f := (-1)^{\tilde{f}+1}\{f,\ \} = (-1)^{\tilde{a}\tilde{f}}\mathfrak{S}^{ab}\frac{\partial f}{\partial x^b}\frac{\partial}{\partial x^a}. \tag{1.3}$$

It has parity opposite to that of f. Notice that

$$X_{\{f,g\}} = [X_f, X_g] \tag{1.4}$$

and

$$X_{fg} = (-1)^{\tilde{f}}fX_g + (-1)^{\tilde{g}+\tilde{f}\tilde{g}}gX_f \tag{1.5}$$

(or $X_{fg} = X_f \circ g + (-1)^{\tilde{f}}fX_g + (-1)^{\tilde{f}}\{f,g\}$, where in the r.h.s. there is the multiplication by g followed by the action of X_f).

We denote by

$$\mathcal{L}_f := \mathcal{L}_{X_f} \tag{1.6}$$

the Lie derivative of geometric objects on M (e.g., tensor fields of a given type) w.r.t. the vector field X_f. The Lie derivative along X_f makes the space of particular geometric objects on M into a "Poisson module" over the odd Poisson algebra $C^\infty(M)$, meaning that

$$[\mathcal{L}_f, g] = (-1)^{\tilde{f}+1}\{f,g\} = X_f g \tag{1.7}$$

$$[\mathcal{L}_f, \mathcal{L}_g] = \mathcal{L}_{\{f,g\}}. \tag{1.8}$$

[1] A collection of useful formulae can be found in the introductory section of [10]

1.2 Odd Laplacian on functions

Select a volume form $\boldsymbol{\rho} = \rho\, Dx$ on an odd Poisson manifold M. We assume that $\boldsymbol{\rho}$ is non-degenerate, i.e., that it constitutes a basis element of the space of volume forms. (Here and in the sequel we skip the question of global existence of such a form and the questions of orientation. It is not difficult to supplement the corresponding details.)

Definition 1.1. The *Laplace operator* acting on functions is defined by the formula

$$\Delta_{\boldsymbol{\rho}} f := \text{“div grad } f\text{”} = \operatorname{div}_{\boldsymbol{\rho}} X_f = \frac{\mathcal{L}_f \boldsymbol{\rho}}{\boldsymbol{\rho}}. \tag{1.9}$$

It depends on the choice of $\boldsymbol{\rho}$. The operator $\Delta_{\boldsymbol{\rho}}$ is odd.

Formula (1.9) was introduced for the first time in 1989 in [4] (formally, only for the symplectic case) as an invariant construction for the “Δ-operator” of Batalin and Vilkovisky [1, 3][2]. See also [5],[9]. A detailed analysis of this construction for the Poisson case, in a very general algebraic setup, is given in [8].

From formula (1.3) for Hamiltonian vector fields and the formula for the divergence

$$\operatorname{div}_{\boldsymbol{\rho}} X = \frac{1}{\rho}\,(-1)^{\tilde{a}(\tilde{X}+1)}\frac{\partial(\rho X^a)}{\partial x^a}$$

one immediately gets a simple “Laplace–Beltrami type” expression for $\Delta_{\boldsymbol{\rho}}$:

$$\Delta_{\boldsymbol{\rho}} f = \frac{1}{\rho}\,\frac{\partial}{\partial x^a}\left(\rho\,\mathfrak{S}^{ab}\frac{\partial f}{\partial x^b}\right) = \mathfrak{S}^{ab}\frac{\partial^2 f}{\partial x^b \partial x^a} + (\text{lower order terms}). \tag{1.10}$$

The operator $-\frac{\hbar^2}{2}\,\Delta_{\boldsymbol{\rho}}$ is a “quantization” of the function $\mathfrak{S}$ on T^*M. Notice that there is a coordinate-free expression similar to (1.2)

$$\Delta_{\boldsymbol{\rho}} f \cdot \varphi_{\boldsymbol{\rho}} = ((\mathfrak{S}, f), \varphi_{\boldsymbol{\rho}}) \tag{1.11}$$

via the canonical *even* Poisson brackets on T^*M, where $\varphi_{\boldsymbol{\rho}}(x,p) = \rho(x)\delta(p)$ is the generalized function on T^*M corresponding to the volume form $\boldsymbol{\rho}$ on M. Related to it is another useful coordinate-free representation of $\Delta_{\boldsymbol{\rho}}$ given by an integral identity

$$\int_M f(\Delta_{\boldsymbol{\rho}}\, g)\,\boldsymbol{\rho} = \int_M \{f, g\}\,\boldsymbol{\rho}, \tag{1.12}$$

[2]As we see it now, the physical meaning requires rather an operator acting on half-densities, i.e., densities of weight $1/2$. Such operators are the main topic of this paper — though we start off from operators on functions.

analogous to the familiar "Green's first integral formula" in Riemannian geometry (without boundary terms). It is valid if f or g is compactly supported inside M. It follows that $\Delta_{\boldsymbol{\rho}}$ is formally self-adjoint with respect to $\boldsymbol{\rho}$.

From the definition of $\Delta_{\boldsymbol{\rho}}$, together with the formula (1.4), one easily obtains the derivation property with respect to the bracket:

$$\Delta_{\boldsymbol{\rho}}\{f,g\} = \{\Delta_{\boldsymbol{\rho}} f, g\} + (-1)^{\tilde{f}+1}\{f, \Delta_{\boldsymbol{\rho}} g\}. \tag{1.13}$$

Using the definition of X_f, this can be rewritten in the commutator form

$$[\Delta_{\boldsymbol{\rho}}, X_f] = -X_{\Delta_{\boldsymbol{\rho}} f}. \tag{1.14}$$

As for the product of functions, the odd Laplace operator $\Delta_{\boldsymbol{\rho}}$ satisfies

$$\Delta_{\boldsymbol{\rho}}(fg) = (\Delta_{\boldsymbol{\rho}} f)g + (-1)^{\tilde{f}} f(\Delta_{\boldsymbol{\rho}} g) + (-1)^{\tilde{f}+1} 2\{f,g\} \tag{1.15}$$

(directly analogous to another "Green's identity"). It follows from formula (1.5) and the equality $\operatorname{div}_{\boldsymbol{\rho}}(fX) = f \operatorname{div}_{\boldsymbol{\rho}} X + (-1)^{\tilde{X}\tilde{f}} Xf$. Recalling the definition of X_f, we rewrite (1.15) in the commutator form

$$[\Delta_{\boldsymbol{\rho}}, f] = 2X_f + \Delta_{\boldsymbol{\rho}} f. \tag{1.16}$$

From here purely algebraically follow the identities

$$\Delta_{\boldsymbol{\rho}}(f^n) = nf^{n-1}\, \Delta_{\boldsymbol{\rho}} f - n(n-1)f^{n-2}\{f,f\}, \tag{1.17}$$

$$\Delta_{\boldsymbol{\rho}}\, e^{kf} = k\Big(\Delta_{\boldsymbol{\rho}} f - k\,\{f,f\}\Big)\, e^{kf}. \tag{1.18}$$

The definition of the Laplace operator $\Delta_{\boldsymbol{\rho}}$ depends on the choice of a basis volume form $\boldsymbol{\rho}$. If we change $\boldsymbol{\rho}$, the odd Laplacian on functions transforms as

$$\Delta_{\boldsymbol{\rho}'} = \Delta_{\boldsymbol{\rho}} + X_\sigma = \Delta_{\boldsymbol{\rho}} - \{\sigma,\ \}, \tag{1.19}$$

where $\boldsymbol{\rho}' = e^\sigma \boldsymbol{\rho}$.

1.3 Symplectic case: a canonical odd Laplacian on half-densities

The case of a nondegenerate bracket, i.e., of an *odd symplectic manifold* M, is distinguished by the existence of local Darboux coordinates, i.e., local coordinates x^i, θ_i (x^i even, θ_i odd) such that the odd bracket has the form $\{\theta_i, x^j\} = -\{x^j, \theta_i\} = \delta_i^j$ and $\{\theta_i, \theta_j\} = \{x^i, x^j\} = 0$. Suppose $\boldsymbol{\rho} = \rho(x,\theta)\, D(x,\theta)$ in such coordinates. Then, clearly,

$$\Delta_{\boldsymbol{\rho}} = \Delta_0 + X_{\ln \rho} = \Delta_0 - \{\ln \rho,\ \}, \tag{1.20}$$

where

$$\Delta_0 = 2\frac{\partial^2}{\partial x^i \partial \theta_i} \tag{1.21}$$

is a "coordinate Δ-operator". Each term in (1.20) is not invariant under changes of coordinates, only the sum being invariant. It is tempting, however, to consider Δ_0 independently. It turns out that though it does not make sense as an operator acting on functions, an analog of Δ_0 is well-defined as a canonical operator on densities of weight 1/2 (half-densities).

Definition 1.2 (see [6, 7])**.** Let $\boldsymbol{s}$ be a half-density. The *canonical odd Laplace operator* Δ acts on $\boldsymbol{s}$ by the formula

$$\Delta\boldsymbol{s} := \left(2\frac{\partial^2 s}{\partial x^i \partial \theta_i}\right) D(x,\theta)^{1/2}, \tag{1.22}$$

where $\boldsymbol{s} = s(x,\theta)\, D(x,\theta)^{1/2}$ in a local Darboux chart.

The operator (1.22) was introduced in [6]. It was proved in [6, 7] that the definition of Δ on half-densities does not depend on the choice of a Darboux chart, thus yielding a well-defined operator, canonical in the sense that it depends only on the odd symplectic structure and does not require any extra data like a volume form (in contrast to the operator $\Delta_{\boldsymbol{\rho}}$ on functions). The role of half-densities is crucial. One can show that an operator defined in Darboux coordinates by a formula similar to (1.22) on arbitrary densities of weight w will be invariant *only* for $w = 1/2$.

The existence of Δ on half-densities is essentially equivalent to the following statement, which can be traced back to Batalin and Vilkovisky [3]:

Lemma 1.1 ("Batalin–Vilkovisky Lemma").

$$\Delta_0 \left(\operatorname{Ber}\frac{\partial x'}{\partial x}\right)^{1/2} = 0 \tag{1.23}$$

for the change of coordinates between two Darboux charts. Here $x = (x^i, \theta_i)$*,* $x' = (x^{i'}, \theta_{i'})$*, and* Δ_0 *corresponds to the "old" coordinate system.*

We do not give a proof here. Proofs and a detailed analysis of the properties of the operator Δ on half-densities can be found in [7]. Further analysis will be given in subsection 2.3.

The exceptional role of the exponent 1/2 in equation (1.23) cannot be detected on the infinitesimal level. It is related with the possibility to "integrate" infinitesimal canonical transformations to finite ones, due to a deep *groupoid property* of the Batalin–Vilkovisky equations we discuss in Section 2.

The following important formulae were obtained in the calculus of half-densities on an odd symplectic manifold [7]:

$$\Delta(f\boldsymbol{s}) = (\Delta_{\boldsymbol{s}^2} f)\boldsymbol{s} + (-1)^{\tilde{f}} f(\Delta \boldsymbol{s}) \tag{1.24}$$

for an arbitrary function f and a non-degenerate half-density $\boldsymbol{s}$ (at the r.h.s. stands the Laplacian on functions with respect to the volume form $\boldsymbol{s}^2$), and

$$\Delta^2_{\boldsymbol{\rho}} f = \left\{\boldsymbol{\rho}^{-1/2}\,\Delta(\boldsymbol{\rho}^{1/2}), f\right\}, \tag{1.25}$$

where on the l.h.s. stands the Laplace operator on functions with respect to the volume form $\boldsymbol{\rho}$ and at the r.h.s. stands the canonical operator on half-densities. Equation (1.25) in particular implies that the square of the odd Laplacian $\Delta_{\boldsymbol{\rho}}$ is a Hamiltonian vector field. Equation (1.24) can be restated as follows:

$$\Delta(f\boldsymbol{s}) = 2\mathcal{L}_f\, \boldsymbol{s} + (-1)^{\tilde{f}} f\, \Delta \boldsymbol{s}, \tag{1.26}$$

now valid for arbitrary half-densities (indeed, for an even half-density $\boldsymbol{s}$, $(\Delta_{\boldsymbol{s}^2} f)\boldsymbol{s} = (\mathcal{L}_f \boldsymbol{s}^2)\boldsymbol{s}^{-1} = 2\mathcal{L}_f\, \boldsymbol{s}$). Notice that equation (1.26) means that

$$[\Delta, f] = 2\mathcal{L}_f, \tag{1.27}$$

(compare with equation (1.16) for functions).

Physical background. By the geometrical meaning of the Batalin–Vilkovisky quantization procedure, the exponential of the "quantum master action" $e^{i\mathbb{S}/\hbar}$ appearing in the quantum master equation $\Delta\, e^{i\mathbb{S}/\hbar} = 0$ is not a scalar, but the coefficient of a density of weight $1/2$ on the extended phase space of fields, ghosts, antifields, antighosts. The Δ-operator in this master equation exactly corresponds to the canonical Laplace operator on half-densities. In the BV-method $e^{i\mathbb{S}/\hbar}$ is integrated over a Lagrangian submanifold; that is due to the fact that half-densities on the phase space correspond to forms on Lagrangian submanifolds [7]. The difference of the BV quantum action (logarithm of a half-density) from a scalar appears in quantum corrections to the scalar classical action. In the usual Feynman integral (without gauge freedom) the exponential of the "quantum action" $e^{iS/\hbar}$ is the coefficient of a volume form, i.e., of a density of weight 1.

2 Odd Laplace operators: main results

2.1 Square of $\Delta_{\boldsymbol{\rho}}$, master groupoid and modular class

In the *even* Poisson case, the analog of formula (1.9) gives an operator of the first order, a Poisson vector field (see [11] and references therein). The reason why there are no terms of the second order is the skew-symmetry of the Poisson tensor, in contrast with the symmetry of $\mathfrak{S}^{ab}$. In fact, it is nothing but the divergence of the Poisson bivector with respect to a chosen volume form. Under a change of the volume form, this Poisson vector field changes by a Hamiltonian vector field. Hence, its class in the Poisson–Lichnerowicz cohomology does not depend on the volume form and is an invariant of even Poisson manifolds (Weinstein's "modular class").

Now we shall see how a similar class arises for odd Poisson (Schouten) manifolds.

Consider the square of the Laplace operator $\Delta_{\boldsymbol{\rho}}$. Applying $\Delta_{\boldsymbol{\rho}}$ to both sides of (1.15) and using the derivation property with respect to the bracket (1.13), after cancellations we arrive at the simple formula

$$\Delta_{\boldsymbol{\rho}}^2(fg) = (\Delta_{\boldsymbol{\rho}}^2 f)g + (-1)^{\tilde{f}} f(\Delta_{\boldsymbol{\rho}}^2 g), \tag{2.1}$$

which shows that $\Delta_{\boldsymbol{\rho}}^2$ is a vector field. Notice that $\Delta_{\boldsymbol{\rho}}$ is of order $\leqslant 2$, so $\Delta_{\boldsymbol{\rho}}^2$ might, in principle, contain terms of order $\leqslant 4$. However, since $\Delta_{\boldsymbol{\rho}}^2 = (1/2)[\Delta_{\boldsymbol{\rho}}, \Delta_{\boldsymbol{\rho}}]$, the terms of order 4 cancel automatically. The cancellation of the terms of order 3 follows from the vanishing of the canonical bracket $(\mathfrak{S}, \mathfrak{S})$, which is the classical limit of $[\Delta_{\boldsymbol{\rho}}, \Delta_{\boldsymbol{\rho}}]$. These simple considerations *a priori* allow to reduce the order of $\Delta_{\boldsymbol{\rho}}^2$ to 2. Remarkably, the actual order is 1.

(In an algebraic setup, the equivalence of Δ being a derivation of the odd bracket and Δ^2 being a derivation of the associative product was explicitly noticed in [8].)

Since $\Delta_{\boldsymbol{\rho}}^2 = (1/2)[\Delta_{\boldsymbol{\rho}}, \Delta_{\boldsymbol{\rho}}]$ and $\Delta_{\boldsymbol{\rho}}$ is a derivation of the odd bracket, we arrive at

Proposition 2.1. *The vector field $\Delta_{\boldsymbol{\rho}}^2$ is a derivation of the odd bracket, i.e., $\Delta_{\boldsymbol{\rho}}^2$ is a Poisson vector field.*

Recall that in the symplectic case it is always Hamiltonian (equation (1.25)).

Definition 2.1. The Poisson vector field $\Delta_{\boldsymbol{\rho}}^2$ will be called *the modular field* of the Schouten manifold M with respect to the volume form $\boldsymbol{\rho}$.

Proposition 2.2. *The modular field $\Delta_{\boldsymbol{\rho}}^2$ preserves the volume form $\boldsymbol{\rho}$.*

Proof. Denote $Y := \Delta_{\boldsymbol{\rho}}^2$. To calculate the Lie derivative $\mathcal{L}_Y\boldsymbol{\rho}$, we can apply "Green's formula" (1.12) (or the self-adjointness of $\Delta_{\boldsymbol{\rho}}$):

$$\int_M g(\mathcal{L}_Y\boldsymbol{\rho}) = -\int_M (Yg)\boldsymbol{\rho} = -\int_M (\Delta_{\boldsymbol{\rho}}^2 g)\boldsymbol{\rho} = -\int_M (\Delta_{\boldsymbol{\rho}}(1)\, \Delta_{\boldsymbol{\rho}}\, g)\boldsymbol{\rho} = 0,$$

for an arbitrary test function g. Hence $\mathcal{L}_Y\boldsymbol{\rho} = 0$. $\square$

Consider now the transformation of $\Delta^2_{\boldsymbol{\rho}}$ under a change of $\boldsymbol{\rho}$.

Theorem 2.1. *If $\boldsymbol{\rho}' = e^{\sigma}\boldsymbol{\rho}$, then*

$$\Delta^2_{\boldsymbol{\rho}'} = \Delta^2_{\boldsymbol{\rho}} - X_{H(\boldsymbol{\rho}',\boldsymbol{\rho})}, \tag{2.2}$$

where

$$H(\boldsymbol{\rho}',\boldsymbol{\rho}) := \Delta_{\boldsymbol{\rho}}\,\sigma - \frac{1}{2}\{\sigma,\sigma\} = 2e^{-\sigma/2}\,\Delta_{\boldsymbol{\rho}}\,e^{\sigma/2}. \tag{2.3}$$

The odd functions $H(\boldsymbol{\rho}',\boldsymbol{\rho})$ satisfy the following "cocycle conditions":

$$H(\boldsymbol{\rho},\boldsymbol{\rho}) = 0, \tag{2.4}$$

$$H(\boldsymbol{\rho}',\boldsymbol{\rho}) + H(\boldsymbol{\rho},\boldsymbol{\rho}') = 0, \tag{2.5}$$

$$H(\boldsymbol{\rho}'',\boldsymbol{\rho}) = H(\boldsymbol{\rho}'',\boldsymbol{\rho}') + H(\boldsymbol{\rho}',\boldsymbol{\rho}). \tag{2.6}$$

Proof. By equation (1.19), we have $\Delta_{\boldsymbol{\rho}'} = \Delta_{\boldsymbol{\rho}} + X_\sigma$. Hence

$$\Delta^2_{\boldsymbol{\rho}'} = (\Delta_{\boldsymbol{\rho}} + X_\sigma)^2 = \Delta^2_{\boldsymbol{\rho}} + [\Delta_{\boldsymbol{\rho}}, X_\sigma] + X_\sigma^2 = \Delta^2_{\boldsymbol{\rho}} - X_{\Delta_{\boldsymbol{\rho}}\sigma} + \frac{1}{2}[X_\sigma, X_\sigma] = \Delta^2_{\boldsymbol{\rho}} - X_{\Delta_{\boldsymbol{\rho}}\sigma - \frac{1}{2}\{\sigma,\sigma\}},$$

which proves (2.2-2.3). The cocycle conditions are checked directly:

$$\begin{aligned} H(\boldsymbol{\rho}'',\boldsymbol{\rho}') + H(\boldsymbol{\rho}',\boldsymbol{\rho}) &= \Delta_{\boldsymbol{\rho}'}\sigma' - \frac{1}{2}\{\sigma',\sigma'\} + \Delta_{\boldsymbol{\rho}}\sigma - \frac{1}{2}\{\sigma,\sigma\} = \\ &\Delta_{\boldsymbol{\rho}}\sigma' - \{\sigma,\sigma'\} - \frac{1}{2}\{\sigma',\sigma'\} + \Delta_{\boldsymbol{\rho}}\sigma - \frac{1}{2}\{\sigma,\sigma\} = \\ &\Delta_{\boldsymbol{\rho}}(\sigma+\sigma') - \frac{1}{2}\{\sigma+\sigma',\sigma+\sigma'\} = H(\boldsymbol{\rho}'',\boldsymbol{\rho}). \end{aligned}$$

Similarly $H(\boldsymbol{\rho},\boldsymbol{\rho}') = -H(\boldsymbol{\rho}',\boldsymbol{\rho})$, and $H(\boldsymbol{\rho},\boldsymbol{\rho}) = 0$ is obvious. □

Remark 2.1. The vector fields $X_{H(\boldsymbol{\rho}',\boldsymbol{\rho})} = \Delta^2_{\boldsymbol{\rho}} - \Delta^2_{\boldsymbol{\rho}'}$ being "coboundaries" trivially satisfy the "cocycle condition". However, one might expect that the corresponding Hamiltonians $H(\boldsymbol{\rho}',\boldsymbol{\rho})$ define a cocycle only up to Casimirs; the key statement is that the cocycle conditions are satisfied exactly.

Consider the factors e^σ as "arrows" between volume forms on M. This is an action of a group on the set of volume forms. Consider now only the arrows $\boldsymbol{\rho} \to \boldsymbol{\rho}' = e^\sigma\boldsymbol{\rho}$ satisfying the equation $\Delta_{\boldsymbol{\rho}}\,e^{\sigma/2} = 0$ (the equation on an arrow depends on its "source"). From formula (2.3) and the cocycle conditions (2.4), (2.5), (2.6) follows an important statement.

Theorem 2.2. *The solutions of the equations* $\Delta_{\boldsymbol{\rho}}\, e^{\sigma/2} = 0$ *form a groupoid. That is: for three volume forms* $\boldsymbol{\rho}, \boldsymbol{\rho}', \boldsymbol{\rho}''$, *if* $\Delta_{\boldsymbol{\rho}}\, e^{\sigma/2} = 0$ *and* $\Delta_{\boldsymbol{\rho}'}\, e^{\tau/2} = 0$, *then* $\Delta_{\boldsymbol{\rho}}\, e^{(\sigma+\tau)/2} = 0$ *("composition"). Also, if* $\Delta_{\boldsymbol{\rho}}\, e^{\sigma/2} = 0$, *then* $\Delta_{\boldsymbol{\rho}'}\, e^{-\sigma/2} = 0$ *("inverses"). Here* $\boldsymbol{\rho}' = e^{\sigma}\boldsymbol{\rho}$, $\boldsymbol{\rho}'' = e^{\tau}\boldsymbol{\rho}'$.

Since the arrows in it satisfy equations that have the form of the quantum "master equation" of the Batalin–Vilkovisky formalism, we shall call the groupoid defined in Theorem 2.2 the *master groupoid* of an odd Poisson manifold M. The space of all non-degenerate volume forms on M is partitioned into orbits of the master groupoid.

The factor $1/2$ in the exponent is exceptional; no such property holds for any $e^{\lambda\sigma}$ other than $\lambda = 1/2$.

Example 2.1. Let M be an odd symplectic manifold. Consider as points of a groupoid all Darboux coordinate systems and as arrows the canonical transformations between them. A homomorphic image of it is the groupoid whose points are coordinate volume forms (in Darboux coordinates) and whose arrows are the respective Jacobians J. Consider a new groupoid with arrows J^{λ}. Infinitesimally, $\Delta_0 J^{\lambda} = 0$ for any λ, where $J = 1 + \varepsilon \operatorname{div}_0 X_F = 1 + \varepsilon\Delta_0 F$, where F is a Hamiltonian generating the canonical transformation, because $\Delta_0^2 = 0$. However, we can glue together the conditions for infinitesimal transformations $\Delta_0 J^{\lambda} = 0$ only when $\lambda = 1/2$. Thus we arrive at the identity $\Delta_0 J^{1/2} = 0$ for a finite transformation, i.e., to the Batalin–Vilkovisky Lemma. That means that all "Darboux coordinate volume forms" belong to the same orbit of the master groupoid, i.e., in the symplectic case the orbit space has a natural base point.

Notice now that by Theorem 2.1 the modular vector field $\Delta_{\boldsymbol{\rho}}^2$ depends on a volume form up to a Hamiltonian vector field. Hence it defines a cohomology class $[\Delta_{\boldsymbol{\rho}}^2]$ depending only on the odd bracket structure. We call $[\Delta_{\boldsymbol{\rho}}^2]$ the *modular class* of the Schouten manifold M. More precisely, consider T^*M with the canonical even bracket. The Schouten tensor (1.1) satisfies $(\mathfrak{S}, \mathfrak{S}) = 0$. Hence the operator $D := \operatorname{ad}\mathfrak{S} = (\mathfrak{S},\ )$ on the space $C^{\infty}(T^*M)$ is an odd differential. We call the complex $(C^{\infty}(T^*M), D)$ or its subcomplex consisting of fiberwise polynomial functions, the *Schouten–Lichnerowicz complex* of M, and its cohomology, the *Schouten–Lichnerowicz cohomology* of M. The modular class belongs to the first cohomology group. For odd symplectic manifolds (constant Schouten structure) this class vanishes by equation (1.25). One can show that it also vanishes for all linear Schouten structures. (Compare with Weinstein's class for the Berezin bracket on $\mathfrak{g}^*$ if $\mathfrak{g}$ is non-unimodular.) Whether there are Schouten structures with a nontrivial modular class, is an open question.

2.2 Laplacians on half-densities (Poisson case)

In the symplectic case the canonical odd Laplacian acting on half-densities satisfies equation (1.27). We shall use this identity to characterize Laplacians acting on half-densities on an arbitrary odd Poisson manifold.

Theorem 2.3. *Given an odd Poisson manifold, a linear differential operator Δ acting on densities of weight w and satisfying the equation*

$$[\Delta, f] = 2\mathcal{L}_f \tag{2.7}$$

for an arbitrary function f, exists if and only if $w = 1/2$. In this case Δ is defined uniquely up to a zeroth-order term. If $\boldsymbol{s} = s\boldsymbol{\rho}^{1/2}$, where $\boldsymbol{\rho}$ is some basis volume form, then

$$\Delta(\boldsymbol{s}) = \Delta(s\boldsymbol{\rho}^{1/2}) = (\Delta_{\boldsymbol{\rho}} s)\boldsymbol{\rho}^{1/2} + C\boldsymbol{s} \tag{2.8}$$

(where C is a function). Here $\Delta_{\boldsymbol{\rho}}$ is the Laplace operator on functions.

Proof. Consider densities of weight w and suppose an operator with the property (2.7) exists. For any two operators satisfying (2.7) their difference commutes with the multiplication by functions, hence is a scalar. To fix this scalar, we can set $\Delta(\boldsymbol{\rho}^w) = 0$ for some chosen volume form $\boldsymbol{\rho}$. Then from (2.7) immediately follows that $\Delta(\boldsymbol{s}) = \Delta(s\boldsymbol{\rho}^w) = 2\mathcal{L}_s\boldsymbol{\rho}^w = 2w\mathcal{L}_s\boldsymbol{\rho}\cdot\boldsymbol{\rho}^{w-1} = 2w(\Delta_{\boldsymbol{\rho}} s)\boldsymbol{\rho}\boldsymbol{\rho}^{w-1} = 2w(\Delta_{\boldsymbol{\rho}} s)\boldsymbol{\rho}^w$. (In particular, for $w = 1/2$ we get exactly $\Delta(\boldsymbol{s}) = (\Delta_{\boldsymbol{\rho}} s)\boldsymbol{\rho}^w$.) However, a direct check shows that the operator defined by this formula satisfies condition (2.7) only for $w = 1/2$. For all other weights the actual commutator contains an extra term. We omit here these calculations. □

Choose a volume form $\boldsymbol{\rho}$ and fix normalization by requiring $\Delta(\boldsymbol{\rho}^{1/2}) = 0$. We arrive at the following definition.

Definition 2.2. The *odd Laplace operator on half-densities on an odd Poisson manifold* is

$$\Delta(\boldsymbol{s}) := \boldsymbol{\rho}^{1/2}\Delta_{\boldsymbol{\rho}}(\boldsymbol{s}\boldsymbol{\rho}^{-1/2}) \tag{2.9}$$

where $\boldsymbol{\rho}$ is some basis volume form.

Thus defined, the Laplace operator Δ on half-densities satisfies the commutator condition (2.7)[3]. The uniqueness implies that varying of $\boldsymbol{\rho}$ in (2.9) changes Δ by a scalar term, in contrast with (1.19) for the Laplacian on functions.

[3] By multiplying Δ by a constant the factor "2" in front of the Lie derivative in (2.7) can be replaced by any number. It is convenient, though, to keep coherent normalization with the Laplacians on functions.

Theorem 2.4. *Under the change of volume form $\boldsymbol{\rho} \to \boldsymbol{\rho}' = e^{\sigma}\boldsymbol{\rho}$, the Laplace operator on half-densities transforms as follows:*

$$\Delta' = \Delta - e^{-\sigma/2}\,\Delta_{\boldsymbol{\rho}}\,e^{\sigma/2} = \Delta - \frac{1}{2}\,H(\boldsymbol{\rho}', \boldsymbol{\rho}), \tag{2.10}$$

where we denote by Δ' the operator corresponding to the volume form $\boldsymbol{\rho}'$. Here $H(\boldsymbol{\rho}', \boldsymbol{\rho})$ is the Hamiltonian 2.3. (Notice that the formula $\Delta' - \Delta = -\frac{1}{2}\,H(\boldsymbol{\rho}', \boldsymbol{\rho})$ immediately proves that $H(\boldsymbol{\rho}', \boldsymbol{\rho})$ is a cocycle.)

Proof. From Theorem 2.3 follows that the operator $\Delta' - \Delta$ commutes with all functions. Thus $\Delta' - \Delta = V(\boldsymbol{\rho}', \boldsymbol{\rho})$ must be a function depending on $\boldsymbol{\rho} and \boldsymbol{\rho}'$. To find it, apply $\Delta' - \Delta$ to ${\boldsymbol{\rho}'}^{1/2}$ and use the normalization condition $\Delta'({\boldsymbol{\rho}'}^{1/2}) = 0$. We obtain that $-\Delta({\boldsymbol{\rho}'}^{1/2}) = V(\boldsymbol{\rho}', \boldsymbol{\rho})\,{\boldsymbol{\rho}'}^{1/2}$, i.e., by the definition of the Laplacian on half-densities, $-(\Delta_{\boldsymbol{\rho}} e^{\sigma/2})\,\boldsymbol{\rho}^{1/2} = V(\boldsymbol{\rho}', \boldsymbol{\rho})\,e^{\sigma/2}\boldsymbol{\rho}^{1/2}$, hence $V(\boldsymbol{\rho}', \boldsymbol{\rho}) = -e^{-\sigma/2}\Delta_{\boldsymbol{\rho}} e^{\sigma/2}$. □

Recall that the condition $\Delta_{\boldsymbol{\rho}}\,e^{\sigma/2} = 0$ specifies the arrows of the master groupoid.

Corollary 2.1. *The odd Laplace operator on half-densities is constant on the orbits of the master groupoid.*

The operator Δ actually depends not on a volume form $\boldsymbol{\rho}$, but only on its orbit under the action of the master groupoid. The situation is drastically different from that for Laplacians on functions and for densities of weight $\neq 1/2$.

The master groupoid, which appeared above in relation with the transformation law of $\Delta^2_{\boldsymbol{\rho}}$ on functions, now directly arises from the transformation law of Δ on half-densities. In terms of operators on half-densities, the defining equation $\Delta_{\boldsymbol{\rho}}\,e^{\sigma/2} = 0$ takes the transparent form $\Delta({\boldsymbol{\rho}'}^{1/2}) = 0$.

2.3 Analysis of the symplectic case

Now we can briefly review the symplectic case. It is distinguished by the existence of Darboux charts. With every such chart one can associate a coordinate volume form $\boldsymbol{\rho} = D(x, \theta)$. Moreover, one can construct a global volume form such that for some Darboux atlas this form coincides with the coordinate volume form in every chart of this atlas. The proof uses some facts about the topology of odd symplectic manifolds [7],[9]. We want to emphasize that *there is no natural volume form preserved by all canonical transformations*, unlike the even case with the Liouville measure. However, all "Darboux coordinate" volume forms are in the same orbit of the master groupoid. This orbit is distinguished and it gives rise to the canonical Laplacian on half-densities (1.22), introduced and studied in [6, 7]. Every other Laplacian on half-densities in symplectic case can be expressed via the canonical operator Δ as $\Delta' = \Delta - \boldsymbol{\rho}^{-1/2}\Delta(\boldsymbol{\rho}^{1/2})$.

For odd symplectic manifolds the modular class vanishes, as follows from (1.25). Every Darboux coordinate volume form provides a representative of this class by the zero vector field: the corresponding Laplacian on functions can be written as (1.21) and its square evidently vanishes. Recall that Weinstein's modular class on even symplectic manifolds vanishes due to Liouville's theorem. The odd case is more delicate, as there is no analog of the Liouville volume form. Instead there is a distinguished class of volume forms, contained in the same master groupoid orbit. Lemma 1.1 (the "Batalin–Vilkovisky Lemma") can be viewed as a replacement of Liouville's theorem.

We can try to estimate how "thick" the orbits of the master groupoid are and how many such orbits there are. Even in the symplectic case this analysis is nontrivial.

Consider the orbit through some Darboux coordinate volume form, i.e., all volume forms $e^{\sigma(x,\theta)}D(x,\theta)$ such that $\Delta_0\, e^{\sigma/2} = 0$. For infinitesimal σ, this reduces to $\Delta_0\sigma = 0$. Such σ correspond to closed differential forms on a Lagrangian submanifold [9], [7] and the dimension of the orbit is infinite.

On a general odd Poisson manifold M, for an arbitrary volume form $\boldsymbol{\rho}_0$ the modular vector field $\Delta^2_{\boldsymbol{\rho}}$ is the same for all points in the orbit of $\boldsymbol{\rho}_0$. So the orbit of $\boldsymbol{\rho}_0$ is contained in the submanifold $\Delta^2_{\boldsymbol{\rho}} = \Delta^2_{\boldsymbol{\rho}_0}$ (fixed Poisson vector field). It makes sense to study orbits in such submanifolds. To estimate their codimension we use Theorem 2.1: $\Delta^2_{\boldsymbol{\rho}} - \Delta^2_{\boldsymbol{\rho}_0} = -X_{H(\boldsymbol{\rho},\boldsymbol{\rho}_0)} = 0$, which implies that $H(\boldsymbol{\rho},\boldsymbol{\rho}_0) = 2e^{-\sigma/2}\Delta_{\boldsymbol{\rho}_0}e^{\sigma/2}$ is an odd Casimir function. Infinitesimally we get that $\Delta_{\boldsymbol{\rho}_0}\sigma$ is an odd Casimir. Hence, the codimension of the orbit of $\boldsymbol{\rho}_0$ in the submanifold $\Delta^2_{\boldsymbol{\rho}} = \text{const}$ equals

$$\dim \frac{\Delta^{-1}_{\boldsymbol{\rho}_0}\{\text{All odd Casimirs}\}}{\Delta^{-1}_{\boldsymbol{\rho}_0}\{0\}} = \dim\{\text{All odd Casimirs}\}\,. \tag{2.11}$$

In the symplectic case Casimirs are just constants, and *the orbits of the master groupoid with the square of the Laplace operator being fixed are parametrized by a single modulus, an odd constant* ν. For example, if $\boldsymbol{\rho}_0$ is a Darboux coordinate volume form, then $\nu = \boldsymbol{\rho}^{-1/2}\Delta\boldsymbol{\rho}^{1/2}$, where Δ is the canonical Laplacian on half-densities (1.22). This follows from the analysis performed in other terms in [7].

2.4 Densities of arbitrary weight

The commutation formula (1.16) for $\Delta_{\boldsymbol{\rho}}$ acting on functions ($w=0$) is more complicated than the commutation relation (2.7) for half-densities. The same is true for the behaviour under a change of volume form. We shall now show that this is the generic case and that $w=1/2$ is an exception.

Consider densities of arbitrary weight w. If we fix a basis volume form $\boldsymbol{\rho}$, then they all have the form $\boldsymbol{s}=s\boldsymbol{\rho}^w$. We define an odd Laplace operator acting on densities of weight w by the formula

$$\Delta_{\boldsymbol{\rho}}\,\boldsymbol{s} := (\Delta_{\boldsymbol{\rho}}\,s)\boldsymbol{\rho}^w. \tag{2.12}$$

At the r.h.s. stands the Laplacian on functions. The notation here emphasizes dependence on $\boldsymbol{\rho}$.

Proposition 2.3. *On densities of weight w the commutator of the Laplacian (2.12) and the multiplication by arbitrary functions is given by the formula*

$$[\Delta_{\boldsymbol{\rho}}, f] = 2\mathcal{L}_f + (1-2w)\,\Delta_{\boldsymbol{\rho}} f. \tag{2.13}$$

Plugging $w=0$ and $w=1/2$ we recover formulae (1.16) and (2.7), respectively. Clearly the case $w=1/2$ is exceptional as the second term in (2.13) vanishes identically.

Suppose that we change the basis volume form $\boldsymbol{\rho}$. What is the transformation law for $\Delta_{\boldsymbol{\rho}}$?

Proposition 2.4. *On densities of weight w the Laplacian $\Delta_{\boldsymbol{\rho}}$ transforms as follows:*

$$\begin{aligned}\Delta_{\boldsymbol{\rho}'} &= \Delta_{\boldsymbol{\rho}} + (1-2w)\mathcal{L}_\sigma - 4w(1-w)\,e^{-\sigma/2}\,\Delta_{\boldsymbol{\rho}}\,e^{\sigma/2}\\ &= \Delta_{\boldsymbol{\rho}} + (1-2w)\mathcal{L}_\sigma - 2w(1-w)\,H(\boldsymbol{\rho}',\boldsymbol{\rho}).\end{aligned} \tag{2.14}$$

Plugging $w=0$ and $w=1/2$ we recover formulae (1.19) and (2.10).

We see again that the case $w=1/2$ is exceptional, because only then the transformation (2.14) involves just a scalar additive term. In all other cases the transformation includes a differential operator of the first order. Functions and volume forms, i.e., $w=0$ and $w=1$, are somewhat special too, because for them this differential operator simplifies to a Lie derivative.

3 Comparison with Riemannian and even Poisson geometry

3.1 Laplacians in Riemannian geometry

Let us look at Riemannian geometry from the perspective of odd Poisson geometry (instead of the converse). Slightly abusing language, by a Riemannian structure we mean a symmetric tensor with upper indices g^{ab} not necessarily invertible. The non-degenerate situation is then an analog of a symplectic structure. Both Riemannian and odd Poisson structures are specified by *quadratic Hamiltonians*, even or odd respectively, which are functions on the same manifold T^*M. For simplicity below we will write all Riemannian formulae only for even manifolds, though everything, of course, works in the super case.

Without assuming that g^{ab} is invertible we can use an arbitrary volume form $\boldsymbol{\rho} = \rho\, d^n x$ to define a Laplace operator on functions:

$$\Delta_{\boldsymbol{\rho}} f := \operatorname{div}_{\boldsymbol{\rho}} \operatorname{grad} f = \frac{1}{\rho}\frac{\partial}{\partial x^a}\left(\rho g^{ab}\frac{\partial f}{\partial x^b}\right). \tag{3.1}$$

The following properties are similar to those of odd Laplacian (1.15), (1.18):

$$\Delta_{\boldsymbol{\rho}}(fg) = (\Delta_{\boldsymbol{\rho}} f)g + f(\Delta_{\boldsymbol{\rho}} g) + 2\langle f, g\rangle, \tag{3.2}$$

$$\Delta_{\boldsymbol{\rho}} e^f = \Big(\Delta_{\boldsymbol{\rho}} f - \langle f, g\rangle\Big)\, e^f, \tag{3.3}$$

where the Poisson bracket is replaced by the "scalar product of gradients" $\langle f, g\rangle = g^{ab}\, \partial_a f\, \partial_b g$. Under changes of volume form the Laplacian (3.1) transforms as

$$\Delta_{\boldsymbol{\rho}'} = \Delta_{\boldsymbol{\rho}} + \operatorname{grad} \sigma, \tag{3.4}$$

where $\boldsymbol{\rho}' = e^{\sigma}\boldsymbol{\rho}$. What fails and has no analogy in the Riemannian case is the formulae involving the action of the Laplacian on the bracket.

On densities of arbitrary weight w we can define the Laplacian by the same formula as (2.12):

$$\Delta_{\boldsymbol{\rho}}\, \boldsymbol{s} := (\Delta_{\boldsymbol{\rho}}\, s)\boldsymbol{\rho}^w, \tag{3.5}$$

if $\boldsymbol{s} = s\boldsymbol{\rho}^w$. It has properties analogous to (2.14) and (2.13).

In particular, the case of half-densities is again distinguished. Analogs of Theorem 2.3 and Theorem 2.4 hold. The Laplace operator on half-densities Δ satisfies the condition

$$[\Delta, f] = 2\mathcal{L}_{\operatorname{grad} f}. \tag{3.6}$$

Under the change of volume form $\boldsymbol{\rho} \to \boldsymbol{\rho}' = e^{\sigma}\boldsymbol{\rho}$ the Laplacian on half-densities transforms as

$$\Delta' = \Delta - e^{-\sigma/2}\, \Delta_{\boldsymbol{\rho}}\, e^{\sigma/2}, \tag{3.7}$$

as in (2.10). We again arrive at a groupoid; the Batalin–Vilkovisky equation is replaced by the Laplace equation $\Delta({\boldsymbol{\rho}'}^{1/2}) = 0$.

Altogether we see that the analogy between odd Poisson and "upper" Riemannian geometry goes unexpectedly far.

3.2 Geometries controlled by a tensor T^{ab}

Odd Poisson geometry has analogies with Riemannian geometry as well as with even Poisson geometry. All three geometries are controlled by a rank 2 tensor, say, T^{ab}. The difference is in the type of symmetry and in the parity of T^{ab}.

In the even Poisson case, $T^{ab} = P^{ab}$, $P^{ab} = -(-1)^{\tilde a \tilde b + \tilde a + \tilde b} P^{ba}$ and $\widetilde{P^{ab}} = \tilde a + \tilde b$. It corresponds to an even bivector field $P \in C^\infty(\Pi T^* M)$. In the odd Poisson (= Schouten) case, $T^{ab} = \mathfrak{S}^{ab}$, $\mathfrak{S}^{ab} = (-1)^{\tilde a \tilde b} \mathfrak{S}^{ba}$, but $\widetilde{\mathfrak{S}^{ab}} = \tilde a + \tilde b + 1$. It corresponds to an odd quadratic Hamiltonian $\mathfrak{S} \in C^\infty(T^* M)$. In the Riemannian case, $T^{ab} = g^{ab}$, $g^{ab} = (-1)^{\tilde a \tilde b} g^{ba}$ and $\widetilde{g^{ab}} = \tilde a + \tilde b$. It corresponds to an even Hamiltonian $H = \frac{1}{2}\, g^{ab} p_b p_a \in C^\infty(T^* M)$. In all cases the structure on M is obtained from the tensor T via the canonical brackets on $T^* M$ or $\Pi T^* M$.

It is convenient to draw a table:

<table>
<tr><td>Even Poisson
$P^{ab} = (-1)^{(\tilde a+1)(\tilde b+1)} P^{ba}$</td><td>Odd Poisson (Schouten)
$\mathfrak{S}^{ab} = (-1)^{\tilde a \tilde b} \mathfrak{S}^{ba}$</td><td>Even Riemannian
$g^{ab} = (-1)^{\tilde a \tilde b} g^{ba}$</td><td>Odd Riemannian
$\chi^{ab} = (-1)^{(\tilde a+1)(\tilde b+1)} \chi^{ba}$</td></tr>
<tr><td>$P = \frac{1}{2} P^{ab}(x) x^*_b x^*_a$
$\tilde P = 0$
$\Pi T^* M$, canonical
Schouten bracket $[\![\ ,\ ]\!]$</td><td>$\mathfrak{S} = \frac{1}{2} \mathfrak{S}^{ab}(x) p_b p_a$
$\tilde{\mathfrak{S}} = 1$
$T^* M$, canonical
Poisson bracket $(\ ,\ )$</td><td>$H = \frac{1}{2} g^{ab}(x) p_b p_a$
$\tilde g = 0$
$T^* M$, canonical
Poisson bracket $(\ ,\ )$</td><td>$\chi = \frac{1}{2} \chi^{ab}(x) x^*_b x^*_a$
$\tilde \chi = 1$
$\Pi T^* M$, canonical
Schouten bracket $[\![\ ,\ ]\!]$</td></tr>
<tr><td>$\{f,g\} = [\![f, [\![P,f]\!]]\!]$

$f \mapsto X_f = \{f,\ \}$

Jacobi for $\{\ ,\ \}$
$\Leftrightarrow [\![P,P]\!] = 0$</td><td>$\{f,g\} = (f, (\mathfrak{S}, g))$

$f \mapsto X_f = (-1)^{\tilde f + 1}\{f,\ \}$

Jacobi for $\{\ ,\ \}$
$\Leftrightarrow (\mathfrak{S}, \mathfrak{S}) = 0$</td><td>$\langle f,g \rangle = (f,(H,g))$
$= (-1)^{\tilde f \tilde a} g^{ab} \frac{\partial f}{\partial x^b} \frac{\partial g}{\partial x^a}$
$f \mapsto \operatorname{grad} f = \langle f,\ \rangle$

None</td><td>$\langle f,g \rangle = [\![f, [\![\chi, f]\!]]\!]$

$f \mapsto \operatorname{grad} f$

None</td></tr>
<tr><td>$\Delta_{\boldsymbol\rho} f = \operatorname{div}_{\boldsymbol\rho} X_f$

1st order
even operator</td><td>$\Delta_{\boldsymbol\rho} f = \operatorname{div}_{\boldsymbol\rho} X_f$

2nd order
odd operator</td><td>$\Delta_{\boldsymbol\rho} f = \operatorname{div}_{\boldsymbol\rho} \operatorname{grad} f$

2nd order
even operator</td><td>$\Delta_{\boldsymbol\rho} f = \operatorname{div}_{\boldsymbol\rho} \operatorname{grad} f$

1st order
odd operator</td></tr>
<tr><td colspan="4">$\boldsymbol\rho' = e^\sigma \boldsymbol\rho$</td></tr>
<tr><td>$\Delta_{\boldsymbol\rho'} = \Delta_{\boldsymbol\rho} + X_\sigma$</td><td>$\Delta_{\boldsymbol\rho'} = \Delta_{\boldsymbol\rho} + X_\sigma$</td><td>$\Delta_{\boldsymbol\rho'} = \Delta_{\boldsymbol\rho} + \operatorname{grad}\sigma$</td><td>$\Delta_{\boldsymbol\rho'} = \Delta_{\boldsymbol\rho} + \operatorname{grad}\sigma$</td></tr>
<tr><td>Modular class: $[\Delta_{\boldsymbol\rho}]$</td><td>Modular class: $[\Delta^2_{\boldsymbol\rho}]$</td><td>None</td><td>None</td></tr>
<tr><td colspan="4">Laplacian on half-densities: $\Delta(\boldsymbol s) = \boldsymbol\rho^{1/2} \Delta_{\boldsymbol\rho}(\boldsymbol s \boldsymbol\rho^{-1/2})$</td></tr>
<tr><td>nothing good</td><td colspan="2">$\Delta' - \Delta = \frac{1}{2} H(\boldsymbol\rho', \boldsymbol\rho)$
$H(\boldsymbol\rho', \boldsymbol\rho) = e^{-\sigma/2}\, \Delta_{\boldsymbol\rho}(e^{\sigma/2})$</td><td>nothing good</td></tr>
<tr><td>None</td><td colspan="2">Master groupoid:
$\boldsymbol\rho \to \boldsymbol\rho' = e^\sigma \boldsymbol\rho$ such that $\Delta_{\boldsymbol\rho}\, e^{\sigma/2} = 0$</td><td>None</td></tr>
</table>

The fourth case included in the table but which we shall not develop here, is that of odd Riemannian structure. Then, $T^{ab} = \chi^{ab}$ corresponds to an odd bivector field. In a non-degenerate situation, the tensor with lower indices corresponding to an odd quadratic Hamiltonian $\mathfrak{S}$ is an odd 2-form and that corresponding to an odd bivector field χ is an odd symmetric tensor.

Let us comment on some of the similarities and differences. Algebraic similarity of even/odd Poisson brackets as well as of even/odd metrics is clear. However, the second order Laplace operator on functions in the odd Poisson and even Riemannian cases corresponds to a vector field in the even Poisson and odd Riemannian cases. Algebraically responsible for arising of the "master groupoid" are formulae like $\Delta_{\boldsymbol{\rho}}\, e^f = e^f\big(\Delta_{\boldsymbol{\rho}}\, f - \{f, f\}\big)$. They do not appear in even Poisson geometry. For the same reason, in it and in odd Riemannian geometry there is no interesting theory of Laplace operators acting on various densities. (Notice that analogies and differences come in pairs in this picture.)

3.3 BV formalism and quantum mechanics

Without going into details we want to point out a geometric analogy between the Batalin–Vilkovisky quantization and ordinary quantum mechanics.

Let us make a simple but important remark: the wave function in the Schrödinger equation is a half-density. Hence the quantum Hamiltonian in the Schrödinger picture is an operator acting on half-densities. Let it have the form $\hat{H} = -\frac{\hbar^2}{2}\Delta + U$, where Δ is the Laplace operator on half-densities. It satisfies the commutator condition (3.6). $-\frac{\hbar^2}{2}\Delta$ is a quantization of the classical "free" Hamiltonian $H_0 = \frac{1}{2}\, g^{ab}p_b p_a$. The quasiclassical solution of the Schrödinger equation is obtained by substituting the wave function as $\boldsymbol{\psi} = e^{iS/\hbar}\boldsymbol{u}$ where $\boldsymbol{u} = \sum (i\hbar)^n \boldsymbol{u}_n$ is a half-density and S is a function independent of $\hbar$. The classical term is $\langle \operatorname{grad} S, \operatorname{grad} S\rangle$. The first semiclassical term is $i\hbar\, \mathcal{L}_{\operatorname{grad} S}\boldsymbol{u}_0$, etc. The commutator formula (3.6) (which is equivalent to the vanishing of the subprincipal symbol) implies that the l.h.s. of the transport equations will contain only the Lie derivative along the gradient of the classical action defined from the classical Hamilton–Jacobi equation.

Likewise, the "quantum master equation" of the Batalin–Vilkovisky quantization procedure (the Batalin–Vilkovisky equation) is the equation $\Delta\boldsymbol{s} = 0$ for a half-density $\boldsymbol{s}$ on an odd symplectic manifold, where Δ is the canonical odd Laplacian. Writing formally $\boldsymbol{s} = e^{iS/\hbar}\boldsymbol{u}$ as above (S is a function, $\boldsymbol{u}$ a half-density) and using the commutator identity (2.7), one gets a similar expansion in $i\hbar$ starting from $\{S, S\} = 0$ (the "classical master equation").

Hence, we have the following analogy: the quantum master equation corresponds to the Schrödinger equation; its solution (a half-density, often formally written in the purely exponential form $e^{i\mathbb{S}/\hbar}$ via the so-called "quantum effective action") corresponds to the wave function; the classical master equation corresponds to the Hamilton–Jacobi equation or the eikonal equation; the next quantum corrections, as in the Schrödinger case, are expressed in terms of Lie derivative along the Hamiltonian vector field $\mathcal{L}_S$, where S is a solution of the classical master equation.

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