On odd Laplace operators. II

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Abstract

We analyze geometry of the second order differential operators, having in
mind applications to Batalin–Vilkovisky formalism in quantum field theory.
As we show, an exhaustive picture can be obtained by considering pencils of
differential operators acting on densities of all weights simultaneously. The
algebra of densities, which we introduce here, has a natural invariant scalar
product. Using it, we prove that there is a one-to-one correspondence be-
tween second-order operators in this algebra and the corresponding brackets.
A bracket on densities incorporates a bracket on functions, an "upper connec-
tion" in the bundle of volume forms, and a term similar to the "Brans–Dicke
field" of the Kaluza–Klein formalism. These results are valid for even opera-
tors on a usual manifold as well as for odd operators on a supermanifold. For
an odd operator Δ we show that conditions on the order of the operator Δ^2
give an hierarchy of properties such as flatness of the upper connection and
the Batalin–Vilkovisky master equation. In particular, we obtain a complete
description of generating operators for an arbitrary odd Poisson bracket.

Contents

1   Introduction

This paper is a direct continuation of our paper [14]. However, it is completely independent and can be read as such.

There are two motivations for this work.

The first motivation comes from the Batalin–Vilkovisky formalism in quantum physics. The problem is to give a description of all the so-called 'Δ-operators'. A particular algebraic problem related to this is to give a description of all generating operators for a given odd Poisson algebra. The second motivation is entirely geometrical. The problem is to describe geometric structures encoded in a differential operator. In particular, the question is, what is necessary to recover a differential operator from its principal symbol.

It is well known that various quantities in quantum field theory can be expressed via the Feynman integral

$$Z = \int e^{\frac{i}{\hbar} S[\varphi]} \, \mathcal{D}\varphi. \tag{1}$$

Here integration is over all field configurations and $S[\varphi]$ stands for the classical action functional. However, if the theory possesses a gauge freedom (like electrodynamics or Yang–Mills theory), a modification is required. According to the most up-to-date comprehensive procedure, — the Batalin–Vilkovisky formalism, — the recipe is as follows [3, 4, 5]. The (infinite-dimensional) manifold of fields φ has to be extended and necessarily becomes a supermanifold possessing an odd symplectic structure. The classical action $S[\varphi]$ and the integral (1) are replaced by an "extended" action $\mathcal{S}[\Phi]$ and by the integral[1]

$$Z_{\mathrm{BV}} = \int e^{\frac{i}{\hbar} \mathcal{S}[\Phi]} \, \mathcal{D}\Phi \,, \tag{2}$$

where integration is over a Lagrangian submanifold. The key condition is that the extended action must satisfy the Batalin–Vilkovisky "quantum master equation"

$$\Delta e^{\frac{i}{\hbar} \mathcal{S}[\Phi]} = 0, \tag{3}$$

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[1]Strictly speaking, there must be $\sqrt{\mathcal{D}\Phi}$ instead of $\mathcal{D}\Phi$, since it is the half-densities that give volume forms on Lagrangian submanifolds [9, 11].

and this secures the gauge invariance of the quantum theory. The Batalin–Vilkovisky Δ-operator is an "odd Laplacian" associated with an odd symplectic structure. We want to emphasize that the precise geometric formulation of the above-said, including the precise definition of the operator Δ, is a non-trivial task, to which a lot of work has been devoted, see [8, 12, 13, 9, 10, 11, 14] and [17, 19, 18]. Initially Δ has been thought of as an operator acting on functions. In [9, 10] it was shown that Δ should be considered as an operator on half-densities (= semidensities) rather than functions. Moreover, as we have shown in [14], the best understanding of Δ can be achieved by considering it on densities of various weights. There is a remarkable similarity between odd Poisson geometry and the usual Riemannian geometry noticed in [14].

The Batalin–Vilkovisky operator "generates" (in a precise sense) the odd symplectic structure of the extended phase space. In a more abstract setup, every 'Δ-operator' on functions generates an odd bracket. A question that remained open, is how to describe all operators generating a given bracket. A lot of work was devoted to this problem, in the geometric as well as in an algebraic setting. (Notice that the Batalin–Vilkovisky quantization formalism motivated the introduction of various algebraic structures, such as the "Batalin–Vilkovisky algebras", the study of which has developed into an independent area.) See in particular [1, 2], [12], [15]. The current paper gives a complete solution of this problem.

A related question is how to obtain an operator on half-densities or densities of any other weight from an operator on functions, or vice versa.

These questions naturally bring us to the second, purely geometrical motivation for the present paper.

Suppose we are given a differential operator Δ of order $\leqslant n$ acting on functions on a manifold M. Which geometric structures are naturally associated with it? We want to stress that we are considering an operator acting on functions on a manifold without any extra structure (like a Riemannian structure) given *a priori*. If we write this operator in local coordinates as

$$\Delta = \sum_{k=0}^{n} \frac{1}{k!} A^{a_1,\ldots,a_k}(x) \, \partial_{a_1} \ldots \partial_{a_k}, \tag{4}$$

the coefficients $A^{a_1,\ldots,a_k}$ are transformed in a complicated way under a change of coordinates. Which geometric information is encoded in them?

First of all, as it is well known, the top order part defines an invariant object, called the principal symbol of Δ:

$$\sigma(\Delta) = \frac{1}{n!} A^{a_1,\ldots,a_n}(x) \, p_{a_1} \ldots p_{a_n}. \tag{5}$$

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The principal symbol $\sigma(\Delta)$ is an invariantly defined function on T^*M. It also can be approached from a purely algebraic viewpoint (see subsection 4.2).

What about the geometric meaning of the lower order terms in (4)? Hörmander has introduced the so-called subprincipal symbol of Δ, which essentially is the principal symbol of $\Delta - (-1)^n \Delta^*$ where the adjoint operator Δ^* (acting on the same space as Δ) depends on a choice of volume element, e.g., given by a coordinate system. Hence the subprincipal symbol $\operatorname{sub}\Delta$ is not a genuine function on T^*M, but has a non-trivial transformation law. In the standard approach this is considered as a nuisance that can be overcome by trading functions for half-densities, for which the adjoint operator is intrinsically defined. However, a closer look at the transformation law of the subprincipal symbol $\operatorname{sub}\Delta$ for operators on functions reveals that it is very similar to a connection in the bundle of volume forms $\operatorname{Vol} M$.

This is quite unexpected as, let us repeat, we have started from an operator acting on scalar functions, with no geometric data (like bundles and connections) being given beforehand. In particular, this prompts to consider operators acting on densities of various weights $w \in \mathbb{R}$ together. As soon as we adopt this viewpoint, the picture immediately clears.

We introduce the *algebra of densities* $\mathfrak{V}(M)$ on a manifold M as the algebra of densities of all weights $w \in \mathbb{R}$ under the tensor multiplication. Formally, it is the algebra of sections of the direct sum of the bundles $|\operatorname{Vol}(M)|^w$ over all $w \in \mathbb{R}$. $\mathfrak{V}(M)$ possesses a natural invariant scalar product, and it contains, in particular, functions, volume forms and half-densities. One can consider differential operators in the algebra $\mathfrak{V}(M)$, and there is a natural notion of the adjoint operator. It is possible to give a nice classification of derivations for $\mathfrak{V}(M)$.

The main results of the present paper can be summarized as follows.

We describe differential operators in $\mathfrak{V}(M)$. A differential operator of order $\leqslant 2$ in $\mathfrak{V}(M)$ is equivalent to a quadratic pencil

$$\Delta_w = \Delta_0 + w\,A + w^2\,B,$$

where Δ_0 is a second-order operator on functions, A and B have order 1 and 0 respectively. We prove that there is a one-to-one correspondence between the self-adjoint operators with the condition $\Delta_0(1){=}0$ and the corresponding brackets in $\mathfrak{V}(M)$. A bracket in $\mathfrak{V}(M)$ from the viewpoint of M is a "long bracket" incorporating a bracket on functions, an "upper connection" in the bundle of volume forms and a term similar to the "Brans–Dicke field" g^{55} of the Kaluza–Klein formalism in the relativity theory. This gives a complete description of the geometric information necessary to recover an operator from the corresponding bracket on functions (= the principal symbol)

and unfolds the relations between operators acting on densities of different weights.

These results are valid for even operators as well as for odd operators on a supermanifold. For odd operators and brackets, analysis can be pursued further, leading to remarkable results that have no analogs in the even case. For an odd operator Δ it is natural to study the operator $\Delta^2 = \frac{1}{2}[\Delta, \Delta]$. We show that conditions on the order of Δ^2 give an hierarchy of properties, including the flatness condition for the upper connection and the Batalin–Vilkovisky master equation.

In particular, this gives a complete description of the generating operators for an arbitrary odd Poisson bracket.

Though we mainly consider operators of order $\leqslant 2$ here, we discuss how our results can be generalized to differential operators of higher order. A generalization is also possible to operators of non-zero weight λ (i.e., those mapping densities of weight w to densities of weight $w + \lambda$).

The structure of the paper is the following.

In Section 2 we study arbitrary operators of the second order on functions and densities, and the corresponding brackets. We interpret the subprincipal symbol as an upper connection. We define the algebra $\mathfrak{V}(M)$ and establish its properties. We prove the main classification theorem giving a $1 - 1$ correspondence between operators and brackets in $\mathfrak{V}(M)$, and consider examples.

In Section 3 we consider odd operators and odd brackets. We study the Jacobi identity for an odd long bracket and conditions on the operator Δ^2 corresponding to various properties of brackets.

In Section 4 we discuss generalizations.

2 Second orders operators and long brackets

2.1 Subprincipal symbol as a connection

Let Δ be a differential operator acting on functions on a manifold M. Its principal symbol is a well-defined function on the cotangent bundle T^*M. Compared to it, the so-called subprincipal symbol sub Δ introduced by Hörmander is not a genuine function on T^*M but depends on a choice of coordinates. (Only for operators acting on half-densities the subprincipal symbol becomes an invariant function.) We want to point out that the transformation law for the subprincipal symbol sub Δ for an operator on functions or on densities of any weight $w \neq \frac{1}{2}$ allows to interpret it as a sort of connection.

Indeed, let Δ be an operator of the second order. (Operators of higher order will be treated later.) To avoid complications with signs let Δ be even

and M be an ordinary manifold. Suppose in local coordinates

$$\Delta = \frac{1}{2} S^{ab} \partial_b \partial_a + T^a \partial_a + R, \tag{6}$$

where $S^{ab} = S^{ba}$. Then (with standard conventions) the principal symbol of Δ is $\operatorname{symb} \Delta = -S^{ab} p_b p_a$, which is an invariant quadratic function on $T^* M$, and the subprincipal symbol in local coordinates is [7]:

$$\operatorname{sub} \Delta = i \left(2 T^a - \partial_b S^{ba} \right) p_a. \tag{7}$$

It is nothing but the principal symbol of $\Delta - \Delta^*$ where Δ^* is a coordinate-dependent adjoint operator, defined using a coordinate volume element Dx. Denote

$$\gamma^a = \partial_b S^{ba} - 2 T^a, \tag{8}$$

so that $\operatorname{sub} \Delta = -i \gamma^a p_a$. (In the sequel we omit the factors i.) Then under a change of coordinates the coefficients γ^a are transformed as follows:

$$\gamma^{a'} = \left(\gamma^a + S^{ab} \partial_b \log J \right) \frac{\partial x^{a'}}{\partial x^a}, \tag{9}$$

where $J = Dx'/Dx$ is the Jacobian. If we assume that the matrix S^{ab} is invertible, so that $S_{ab} S^{bc} = \delta_a{}^c$, then it is possible to lower indices and get $\gamma_a = S_{ab} \gamma^b$ with the transformation law

$$\gamma_{a'} = (\gamma_a + \partial_a \log J) \frac{\partial x^a}{\partial x^{a'}}. \tag{10}$$

This is the transformation law for the coefficients of a connection in the bundle of volume forms on M. We see that *the subprincipal symbol of an operator of the second order acting on functions defines a connection in* $\operatorname{Vol} M$, with

$$\nabla_a \rho = (\partial_a + \gamma_a) \rho, \tag{11}$$

$\gamma_a = S_{ab}(2 T^a - \partial_b S^{ba})$, if the "upper metric" S^{ab} given by the principal symbol is non-degenerate. In general, the subprincipal symbol of Δ defines a so-called *upper connection* or "contravariant derivative" in the bundle $\operatorname{Vol} M$:

$$\nabla^a \rho = (S^{ab} \partial_b + \gamma^a) \rho, \tag{12}$$

over the map $S^{\#} \colon T^* M \to T M$ defined by the principal symbol.

"Upper connections" or "contravariant derivatives" were considered earlier, in particular, in the context of Poisson geometry. A natural framework for them is that of Lie algebroids (vector bundles with a Lie bracket of sections and a Lie homomorphism of sections to vector fields). However, for

our purposes this framework is not entirely suitable, since in general we do not expect to have a Lie bracket of covector fields, — except for the case of odd Poisson geometry, see later. Instead, we shall use as an appropriate formalism the following language of "long brackets". ("Long" is meant to remind "long derivatives", the physicists' term for covariant derivatives.)

Fix a bi-derivation in $C^\infty(M)$, denoted as a bracket $\{\ ,\ \}$. We do not assume the Jacobi identity. Let $E \to M$ be a vector bundle. A *long bracket* between functions and sections of E over a given bracket of functions is a bilinear operation $(f, \boldsymbol{s}) \mapsto \{f; \boldsymbol{s}\}$, where $f \in C^\infty(M)$ is a function and $\boldsymbol{s} \in C^\infty(M, E)$ is a section, taking values in sections of E, with the properties:

$$\{fg; \boldsymbol{s}\} = f\{g; \boldsymbol{s}\} + \{f; \boldsymbol{s}\}g, \tag{13}$$

$$\{f; g\boldsymbol{s}\} = \{f, g\}\boldsymbol{s} + g\{f; \boldsymbol{s}\}. \tag{14}$$

Equation (13) means that the value of $\{f; \boldsymbol{s}\}$ depends only on df. A long bracket is related with an "upper connection" by the formula $\nabla^{df}\boldsymbol{s} := \{f; \boldsymbol{s}\}$.

With an operator on functions Δ given by formula (6) we can associate the following bracket of functions and a long bracket over it:

$$\{f, g\} := S^{ab}\partial_b f\, \partial_a g \tag{15}$$

$$\{f; \boldsymbol{\rho}\} := \left(S^{ab}\partial_b f\, \partial_a \rho + \gamma^b \partial_b f\, \rho\right) Dx, \tag{16}$$

where $\boldsymbol{\rho} = \rho Dx$ is a volume form. Here γ^a are given by (8). (If S^{ab} is a metric, then the bracket $\{f, g\}$ is the scalar product of gradients.) An alternative coordinate-free expression is as follows:

$$\{f, g\} := \Delta(fg) - \Delta f\, g - f\, \Delta g + \Delta(1)\, fg, \tag{17}$$

$$\{f; \boldsymbol{\rho}\} := \Delta(f\boldsymbol{\rho}) - \Delta f\, \boldsymbol{\rho} - f\, \Delta\boldsymbol{\rho} + \Delta(1)\, f\boldsymbol{\rho}, \tag{18}$$

where in (18) we define Δ on volume forms as the adjoined operator Δ^*. A direct check shows that (17,18) give (15),(16) and (8); in particular, the coordinate-free formulae (17,18) yield a proof of the transformation law (9).

Let us summarize.

Any second order differential operator Δ acting on functions defines a bracket of functions (17) — essentially, the "polarized" principal symbol of Δ — and an "upper connection" (8) in the bundle of volume forms $\mathrm{Vol}\, M$ — essentially, the subprincipal symbol of Δ — which may be written as a "long bracket" (18) between functions and volume forms extending the bracket of functions.

Hence, starting from operators acting on functions, we are naturally prompted to consider densities. Moreover, the long bracket $\{f; \boldsymbol{\rho}\}$ defined above can be extended by a "Leibniz rule" from densities of weight 1 (volume

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forms) to densities of arbitrary weight. For our purposes we shall need a further generalization with the first argument also replaced by a density. As we shall see below, this will give a "completion" of the theory: after extending both operators and brackets to arbitrary densities it will become possible to establish a one-to-one correspondence between them.

2.2 The algebra of densities

Let M be a supermanifold. Consider densities of arbitrary weights $w \in \mathbb{R}$. (To avoid complications, we can assume that all appropriate Berezinians between local charts are positive.) Under the tensor product densities form a graded commutative algebra: $\boldsymbol{\psi\chi} = (-1)^{\tilde{\psi}\tilde{\chi}}\boldsymbol{\chi\psi}$, $w(\boldsymbol{\psi\chi}) = w(\boldsymbol{\psi}) + w(\boldsymbol{\chi})$. Tilde stands for parity, w for weight; we drop $\otimes$ from the notation. Denote the algebra of densities on M by $\mathfrak{V}(M)$.

The commutative algebra $\mathfrak{V}(M)$ can, in fact, be identified with a certain algebra of functions on an extended manifold $\hat{M} := (\mathrm{Ber}\, TM) \setminus M$, i.e., the frame bundle of the determinant bundle $\mathrm{Ber}\, TM$. The natural coordinates on $\hat{M}$ induced by local coordinates x^a on M are x^a, t where t can be identified with the volume element Dx. A formal sum of densities of various weights $\boldsymbol{\psi} = \sum \psi_w(x)(Dx)^w \in \mathfrak{V}(M)$ can be identified with its "generating function" $\boldsymbol{\psi}(x,t) = \sum \psi_w(x)t^w$ (summation over a finite number of weights). In the sequel we shall use elements of $\mathfrak{V}(M)$ and the corresponding functions on $\hat{M}$ interchangeably.

The algebra of densities $\mathfrak{V}(M)$ has a natural bilinear scalar product:

$$\langle \boldsymbol{\psi}, \boldsymbol{\chi} \rangle := \int_M \psi\chi Dx \tag{19}$$

if $w(\boldsymbol{\psi})+w(\boldsymbol{\chi}) = 1$; otherwise the scalar product is zero. In terms of functions on $\hat{M}$ the scalar product can be expressed as

$$\langle \boldsymbol{\psi}, \boldsymbol{\chi} \rangle := \int_M \mathrm{Res}\left(t^{-2}\,\boldsymbol{\psi}(x,t)\,\boldsymbol{\chi}(x,t)\right)Dx, \tag{20}$$

where Res stands for the residue at $t = 0$. Notice that this algebra possesses a unit. The scalar product satisfies the invariance condition

$$\langle \boldsymbol{\psi\chi}, \boldsymbol{\varphi} \rangle = \langle \boldsymbol{\psi}, \boldsymbol{\chi\varphi} \rangle.$$

One can consider (formally) adjoint operators w.r.t. the scalar product (19). In particular, one has $t^* = t$, $\partial_t^* = -\partial_t + 2t^{-1}$, $\partial_a^* = -\partial_a$; all functions of x^a are self-adjoint. We have $\hat{w}^* = 1 - \hat{w}$.

Let us describe $\operatorname{Der} \mathfrak{V}(M)$, i.e. vector fields on $\hat{M}$. A vector field on $\hat{M}$ of weight λ has the form

$$\boldsymbol{X} = t^\lambda \left(X^a(x) \frac{\partial}{\partial x^a} + X_0(x) \, t \frac{\partial}{\partial t} \right), \tag{21}$$

or

$$\boldsymbol{X} = t^\lambda \big(X^a(x) \, \partial_a + X_0(x) \, \hat{w} \big). \tag{22}$$

The first term has the meaning of a vector density of weight λ on M, i.e., a derivation taking functions to densities of weight λ (this is just the restriction of $\boldsymbol{X}$ to functions on M). The second term has no independent meaning if $X^a \neq 0$; notice the transformation law

$$X^{a'} = J^{-\lambda} X^a \frac{\partial x^{a'}}{\partial x^a}$$
$$X_0' = J^{-\lambda} \left(X_0 + X^a \partial_a \log J \right).$$

There is a canonical operation div (the divergence) on vector fields on $\hat{M}$. This is no wonder, since there is an invariant scalar product of functions on $\hat{M}$, in other words a generalized volume form. The explicit formula for the divergence is

$$\operatorname{div} \boldsymbol{X} = t^\lambda \Big(\partial_a X^a (-1)^{\tilde{a}(\tilde{X}+1)} + (\lambda - 1) \, X_0 \Big), \tag{23}$$

if $\boldsymbol{X}$ is given by (21), (22).

Theorem 2.1. *For $\lambda \neq 1$, every derivation of weight λ in the algebra of densities $\mathfrak{V}(M)$ can be uniquely decomposed into the sum of a divergence-free derivation and a derivation of the form $t^\lambda f(x)\hat{w}$; every divergence-free derivation has the form*

$$\boldsymbol{X} = t^\lambda \left(X^a(x) \, \partial_a - \frac{1}{\lambda - 1} \, \partial_a X^a (-1)^{\tilde{a}(\tilde{X}+1)} \, \hat{w} \right) \tag{24}$$

and is uniquely defined by a vector density $X = (Dx)^\lambda X^a \, \partial_a$ on M. The decomposition of a general $\boldsymbol{X}$ is

$$\boldsymbol{X} = \underbrace{t^\lambda \left(X^a(x) \, \partial_a - \frac{1}{\lambda - 1} \, \partial_a X^a (-1)^{\tilde{a}(\tilde{X}+1)} \, \hat{w} \right)}_{\text{divergence-free part}} + \frac{1}{\lambda - 1} \, \operatorname{div} \boldsymbol{X} \, \hat{w}. \tag{25}$$

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For $\lambda = 0$ the derivation (24) is the Lie derivative along a vector field $X = X^a(x)\,\partial_a$ on M acting on densities. In general, the divergence-free derivation (24) can be seen as a "generalized Lie derivative" along the vector density $X = (Dx)^\lambda X^a\,\partial_a$.

Part of what we are doing is of a purely algebraic nature and holds for arbitrary commutative associative algebras with a unit and with an invariant (non-degenerate) scalar product. In such an algebra, if an operator Δ is of order k, in the algebraic sense, then the adjoint operator Δ^* is also of order k; the difference $\Delta - (-1)^k \Delta^*$ is of order $k-1$. For derivations of such an algebra there exists a canonical divergence.

Recall that in any commutative associative algebra A an abstract divergence operator is a linear map $\mathrm{div}\colon \mathrm{Der}\,A \to A$ with the property

$$\mathrm{div}(aX) = a\,\mathrm{div}\,X + (-1)^{\tilde{a}\tilde{X}}X(a) \tag{26}$$

(see [16], [15]). For functions on a manifold M an abstract divergence and a connection in $\mathrm{Vol}\,M$ are equivalent notions. Indeed, such an equivalence is established by the formula

$$\int_M (\mathrm{div}\,X)\,\boldsymbol{\rho} = -\int_M \nabla_X \boldsymbol{\rho}, \tag{27}$$

and the property (26) translates into the characteristic property of a covariant derivative. (Of course, this can be transported into a more abstract setting.) The explicit formulae:

$$\nabla_a \rho = (\partial_a + \gamma_a)\,\rho, \quad \mathrm{div}\,X = (\partial_a - \gamma_a)X^a(-1)^{\tilde{a}(\tilde{X}+1)}.$$

For abstract divergence operators there is a notion of "curvature" (see [15]). This is exactly the curvature of the corresponding connection on volume forms.

Remark 2.1. Second order differential operators on functions for which the associated upper connections come from genuine connections as $\gamma^a = S^{ab}\gamma_b$ are exactly the "Laplacians"

$$\Delta f = \frac{1}{2}\,(\partial_a - \gamma_a)\left(S^{ab}\partial_b f\right) = \frac{1}{2}\,\mathrm{div}_\gamma \,\mathrm{grad}\,f,$$

where $\mathrm{grad}\,f = S^{ab}\partial_b f$ and div_γ is the divergence operator corresponding to the connection γ_a.

If in the algebra A there is an invariant scalar product, we can define a canonical operation div by either of the equivalent formulae: for $X \in \mathrm{Der}\,A$,

$$\langle \mathrm{div}\,X, a\rangle = -\langle 1, X(a)\rangle, \tag{28}$$

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or

$$\operatorname{div} X = -(X + X^*). \tag{29}$$

Here X^* stands for the operator formally adjoint to X. Notice that $X + X^*$ is of order 0, i.e., an element of A. Immediately checked is that div is an even, linear operation satisfying (26). Also, an identity

$$\operatorname{div}[X, Y] = X(\operatorname{div} Y) + (-1)^{\tilde{X}\tilde{Y}} Y(\operatorname{div} X), \tag{30}$$

holds, meaning that the curvature of the operator div is zero.

Example 2.1. For derivations of $\mathfrak{V}(M)$ we easily get from (28), (29) the explicit formula (23).

Remark 2.2. Formula (29) is very close to the usual definition of a sub-principal symbol. Define div acting on arbitrary operators of order $\leqslant k$ and taking them to operators of order $\leqslant k - 1$ as $\operatorname{div} \Delta := -(\Delta - (-1)^k \Delta^*)$. Then it easily follows that

$$\operatorname{div}[\Delta_1, \Delta_2] = [\operatorname{div} \Delta_1, \Delta_2] + [\Delta_1, \operatorname{div} \Delta_2] + [\operatorname{div} \Delta_1, \operatorname{div} \Delta_2],$$

which implies (30) for derivations. In this abstract setting, the subprincipal symbol of Δ can be defined as the principal symbol of $\operatorname{div} \Delta$, i.e., as the class of $\operatorname{div} \Delta$ modulo operators of order $\leqslant k - 2$. Notice also that $\operatorname{div}^2 = 0$, so we are getting a complex.

2.3 Equivalence between operators and brackets

In this section we shall prove that any bracket on $\hat{M}$, i.e., a bracket in the algebra $\mathfrak{V}(M)$, is in a $1-1$ correspondence with a differential operator of the second order in $\mathfrak{V}(M)$. This should be compared with the fact that a bracket of functions gives only the principal symbol of the generating operator and does not allow to recover it in full.

In any algebra (commutative associative with a unit) by a *bracket* we mean a symmetric bi-derivation:

$$\{a, b\} = (-1)^{\tilde{a}\tilde{b}} \{b, a\} \tag{31}$$

$$\{ka, b\} = (-1)^{\tilde{k}\varepsilon} k\{a, b\} \tag{32}$$

$$\{a, bc\} = \{a, b\}c + (-1)^{(\tilde{a}+\varepsilon)\tilde{b}} b\{a, c\}. \tag{33}$$

Here $\varepsilon \in \mathbb{Z}_2$ is the parity of the bracket. Let us emphasize that in the context of this paper we consider symmetric brackets and hence exclude the usual (antisymmetric even) Poisson brackets.

Notice that from (31-33) follows $\{ab, c\} = (-1)^{\tilde{a}\varepsilon} a\{b, c\} + (-1)^{\tilde{b}\tilde{c}} \{a, c\}b$. In our notation the parity ε "sits" at the opening bracket.

Definition 2.1. A *long bracket* on M is a bracket in the algebra $\mathfrak{V}(M)$. Notation: $\{\boldsymbol{\psi}; \boldsymbol{\chi}\}$.

We use semi-colon for a long bracket, reserving comma for a bracket of functions. A long bracket has weight $\lambda \in \mathbb{R}$ if $w(\{\boldsymbol{\psi}; \boldsymbol{\chi}\}) = w(\boldsymbol{\psi}) + w(\boldsymbol{\chi}) + \lambda$. For $\lambda = 0$ we have a generalization of "long brackets" $\{f; \boldsymbol{s}\}$ considered in section 2.1.

Since a long bracket on M is a usual bracket from the viewpoint of $\hat{M}$, it can be specified by a master Hamiltonian $\hat{S} \in C^\infty(T^*\hat{M})$ as

$$\{\boldsymbol{\psi}; \boldsymbol{\chi}\} = ((\hat{S}, \boldsymbol{\psi}), \boldsymbol{\chi}) \tag{34}$$

(see [14]). The parentheses denote the canonical Poisson bracket on the cotangent bundle. For a bracket of parity ε and weight λ the master Hamiltonian has to be of the form

$$\hat{S} = t^\lambda \frac{1}{2} \left(S^{ab} p_b p_a + 2t\gamma^a p_a p_t + t^2 \theta p_t^2 \right), \tag{35}$$

where p_a and p_t are the momenta conjugate to x^a and t respectively. $\hat{S}$ is of parity ε. That means that for a long bracket we have the following expression:

$$\{\boldsymbol{\psi}; \boldsymbol{\chi}\} = t^\lambda \left(S^{ab} \frac{\partial \boldsymbol{\psi}}{\partial x^b} \frac{\partial \boldsymbol{\chi}}{\partial x^a} (-1)^{\tilde{a}\tilde{\psi}} \right.$$
$$\left. + t\gamma^a \left(\frac{\partial \boldsymbol{\psi}}{\partial x^a} \frac{\partial \boldsymbol{\chi}}{\partial t} + (-1)^{\tilde{a}\tilde{\psi}} \frac{\partial \boldsymbol{\psi}}{\partial t} \frac{\partial \boldsymbol{\chi}}{\partial x^a} \right) + t^2 \theta \frac{\partial \boldsymbol{\psi}}{\partial t} \frac{\partial \boldsymbol{\chi}}{\partial t} \right). \tag{36}$$

Notice that $\hat{w} = t\partial_t$ is the weight operator taking eigenvalues w on densities of weight w. Formula (36) can be rewritten using $\hat{w}$ applied to $\boldsymbol{\psi}$ and $\boldsymbol{\chi}$. Taking off the hats, we come to an equivalent description of the long bracket (36) as a "double pencil" of brackets $\{ \ ; \ \}_{w_1, w_2}$:

$$\{\psi(Dx)^{w_1}; \chi(Dx)^{w_2}\}_{w_1, w_2} = \left(S^{ab} \partial_b \psi \partial_a \chi (-1)^{\tilde{a}\tilde{\psi}} \right.$$
$$\left. + \gamma^a \left(w_2 \partial_a \psi \, \chi + (-1)^{\tilde{a}\tilde{\psi}} w_1 \, \psi \, \partial_a \chi \right) + w_1 w_2 \, \theta \, \psi \chi \right) (Dx)^{w_1 + w_2 + \lambda}. \tag{37}$$

We shall often suppress the subscripts w_1, w_2 in the notation. Equivalently,

$$\{x^a; x^b\} = S^{ab}(Dx)^\lambda$$
$$\{x^a; Dx\} = \gamma^a(Dx)^{\lambda+1}$$
$$\{Dx; Dx\} = \theta(Dx)^{\lambda+2}.$$

It is useful to have the formulae for the transformation law of the coefficients of a long bracket under a change of coordinates. One can deduce that

$$S^{a'b'} = J^{-\lambda} \, S^{ab} \, \frac{\partial x^{b'}}{\partial x^b} \frac{\partial x^{a'}}{\partial x^a} \, (-1)^{\tilde{b}'(\tilde{a}'+\tilde{a})} \tag{38}$$

$$\gamma^{a'} = J^{-\lambda} \left(\gamma^a + S^{ab} \, \partial_b \log J\right) \frac{\partial x^{a'}}{\partial x^a} \tag{39}$$

$$\theta' = J^{-\lambda} \left(\theta + 2\gamma^a \partial_a \log J + S^{ab} \, \partial_b \log J \, \partial_a \log J\right) \tag{40}$$

where $J = Dx'/Dx$ is the Jacobian (Berezinian). We shall be mainly interested in the case $\lambda = 0$. Then it follows that a long bracket incorporates a bracket of functions as well as an "upper connection" γ^a. The space of all long brackets is a vector space. For a fixed bracket of functions given by S^{ab}, upper connections γ^a form an affine space associated with the vector space of vector fields on M. Similarly, for S^{ab}, γ^a fixed, the coefficients θ make up an affine space associated with the space of functions on M. See examples later.

Remark 2.3. If we write the components of the tensor on $\hat{M}$ specifying a long bracket as a block matrix, then

$$(\hat{S}^{\hat{a}\hat{b}}) = \begin{pmatrix} t^\lambda S^{ab} & t^{\lambda+1}\gamma^a \\ t^{\lambda+1}\gamma^a & t^{\lambda+2}\theta \end{pmatrix} \tag{41}$$

and we can see a straightforward analogy with the Kaluza–Klein formalism in field theory, where a metric tensor in a 5-dimensional spacetime combines the usual metric tensor together with a gauge field and an extra scalar field (the "Brans–Dicke field" of tensor-scalar theories of gravitation).

Now we are going to formulate the central theorem of this paper. It has an abstract algebraic counterpart, which is almost trivial if properly stated.

Let A be a commutative associative algebra with a unit. Consider an operator Δ in A of parity ε and introduce an operation $\{a, b\}$ by the formula

$$\{a, b\} = \Delta(ab) - \Delta a \, b - (-1)^{\tilde{a}\varepsilon} a \, \Delta b + \Delta(1) \, ab. \tag{42}$$

Clearly, this operation has parity ε, is bilinear and symmetric: $\{a, b\} = (-1)^{\tilde{a}\tilde{b}}\{b, a\}$, $\{ka, b\} = (-1)^{\tilde{k}\varepsilon}k\{a, b\}$. We shall call Δ a *generating operator* for (42).

Proposition 2.1.

(1) *The operation* (42) *is a bracket, i.e., satisfies* (33)*, if and only if* Δ *is of order* $\leqslant 2$*, in the algebraic sense. Two operators generate the same bracket if and only if they differ by an operator of the first order.*

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(2) *Suppose that the algebra A possesses an invariant scalar product. Given a bracket in A, a generating operator for it is uniquely determined by the extra conditions $\Delta^* = \Delta$ and $\Delta(1) = 0$.*

The first statement is essentially due to Koszul [16]. Recall that $\operatorname{ord} \Delta \leqslant 2$ if all the triple commutators $[[[\Delta, a], b], c]$ vanish. The derivation property of (42) w.r.t. each argument turns out to be equivalent to this condition. As for the second statement, a proof is straightforward: let Δ' be an arbitrary operator generating a given bracket, then one can check that $\Delta := \Delta'' - \Delta''(1)$, where $\Delta'' := \frac{1}{2}(\Delta' + \Delta'^*)$, is a generating operator and it is the only generating operator satisfying the conditions $\Delta^* = \Delta$ and $\Delta(1) = 0$.

For the algebra of densities $\mathfrak{V}(M)$ we can find the generating operator explicitly.

What is an operator of the second order in the algebra $\mathfrak{V}(M)$? It is convenient to use the language of the extended manifold $\hat{M}$ and then translate back using the weight operator $\hat{w}$. Every operator in $\mathfrak{V}(M)$ is equivalent to a pencil of operators Δ_w acting on w-densities (and mapping them to $(w + \lambda)$-densities for operators/pencils of weight λ).

Lemma 2.1. *An operator of order $\leqslant 2$ in the algebra $\mathfrak{V}(M)$ is equivalent to a quadratic pencil of the form*

$$\Delta_w = \Delta_0 + wA + w^2 B, \tag{43}$$

where Δ_0 is an operator of order $\leqslant 2$ acting on functions, A and B are operators of order $\leqslant 1$ and 0 respectively. (Note that A, B do not make sense independently of Δ_0.)

In the language of pencils, the *adjoint pencil* for any pencil Δ_w, corresponding to the adjoint of the operator in $\mathfrak{V}(M)$, is given by $(\Delta_{1-\lambda-w})^*$, because $\hat{w}^* = 1 - \hat{w}$.

Theorem 2.2. *For a given long bracket on M, there exists a unique operator Δ on $\mathfrak{V}(M)$ that satisfies*

$$\Delta^* = \Delta, \quad \Delta(1) = 0, \tag{44}$$

i.e., $\Delta_w^* = \Delta_{1-w}$, $\Delta_0(1) = 0$, and generates the bracket by the formula

$$\{\psi; \chi\} = \Delta(\psi\chi) - \Delta\psi \cdot \chi - (-1)^{\tilde{\psi}\varepsilon}\psi \cdot \Delta\chi. \tag{45}$$

If the bracket is given by (36) or (37), then Δ is given by

$$\Delta = t^\lambda \frac{1}{2} \left(S^{ab}\partial_b\partial_a + \left(\partial_b S^{ba}(-1)^{\tilde{b}(\varepsilon+1)} + (2\hat{w} + \lambda - 1)\gamma^a\right)\partial_a + \right.$$

$$\left. \hat{w}\,\partial_a\gamma^a(-1)^{\tilde{a}(\varepsilon+1)} + \hat{w}(\hat{w} + \lambda - 1)\,\theta\right). \quad (46)$$

Rewriting formula (46) as an operator pencil, we get:

$$\Delta_w = \frac{1}{2} \left(S^{ab}\partial_b\partial_a + \left(\partial_b S^{ba}(-1)^{\tilde{b}(\varepsilon+1)} + (2w + \lambda - 1)\gamma^a\right)\partial_a + \right.$$

$$\left. w\,\partial_a\gamma^a(-1)^{\tilde{a}(\varepsilon+1)} + w(w + \lambda - 1)\,\theta\right). \quad (47)$$

We call (47) the *canonical pencil* corresponding to the long bracket (36, 37).

Consider three important "values" of this pencil (where we set $\lambda = 0$). The operator acting on functions ($w = 0$)

$$\Delta_0 = \frac{1}{2} \left(S^{ab}\partial_b\partial_a + \left(\partial_b S^{ba}(-1)^{\tilde{b}(\varepsilon+1)} - \gamma^a\right)\partial_a\right); \quad (48)$$

the operator acting on volume forms ($w = 1$)

$$\Delta_1 = \frac{1}{2} \left(S^{ab}\partial_b\partial_a + \left(\partial_b S^{ba}(-1)^{\tilde{b}(\varepsilon+1)} + \gamma^a\right)\partial_a + \partial_a\gamma^a(-1)^{\tilde{a}(\varepsilon+1)}\right); \quad (49)$$

the operator acting on half-densities ($w = \frac{1}{2}$)

$$\Delta_{1/2} = \frac{1}{2} \left(S^{ab}\partial_b\partial_a + \partial_b S^{ba}(-1)^{\tilde{b}(\varepsilon+1)}\partial_a + \frac{1}{2}\partial_a\gamma^a(-1)^{\tilde{a}(\varepsilon+1)} - \frac{1}{4}\theta\right). \quad (50)$$

All of them have the same principal symbol defined by the bracket on functions. Notice that operators acting on functions and on volume forms do not depend on the coefficient θ. Given the principal symbol, they are completely defined by the "upper connection" γ^a. The operator on half-densities, on the other hand, depends on γ^a only via $\partial_a\gamma^a(-1)^{\tilde{a}(\varepsilon+1)}$; instead, it includes θ in its zeroth-order term. Neither operator allows to recover the long bracket by itself; knowing the two Δ_0 and $\Delta_{1/2}$ or Δ_1 and $\Delta_{1/2}$ is sufficient. Outside of these exceptional cases it is sufficient to know any single operator Δ_{w_0} to recover γ^a and θ, i.e. the whole pencil Δ_w (see Theorem 2.5 below).

Thus a self-adjoint operator of the second order (in the algebraic sense) on densities, vanishing on 1, contains and is completely defined by the following

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data: a bracket on functions, an "upper connection" γ^a on volume forms, and a quantity θ analogous to the Brans–Dicke field. We shall elucidate the meaning of this quantity in the examples below.

Theorem 2.2 can be proved by a direct calculation starting from a general expression for a pencil Δ_w with indeterminate coefficients depending on w. The property (43) will come about automatically.

There is an alternative approach showing that the generating operator for a given long bracket is nothing but the Laplace operator "div grad". Though this adds nothing in terms of explicit formulae, it will be useful for deducing further properties of Δ.

Let us for a moment come back to a general algebraic situation. Let A be an algebra with a scalar product as above. Consider a bracket in A. Define $\operatorname{grad} a := \{a, \ \}$. This is a derivation of A of parity $\tilde{a} + \varepsilon$. The operator $\operatorname{grad}\colon A \to \operatorname{Der} A$ is linear of parity ε. We have

$$\operatorname{grad}(ab) = (-1)^{\varepsilon\tilde{a}} a \,(\operatorname{grad} b) + (-1)^{(\tilde{a}+\varepsilon)\tilde{b}} b \,(\operatorname{grad} a). \tag{51}$$

We can introduce the *Laplace operator* in the algebra A, corresponding to a given bracket, as

$$\Delta a := \operatorname{div} \operatorname{grad} a. \tag{52}$$

It is an operator of parity ε.

Proposition 2.2. *The Laplace operator satisfies* $\Delta(1) = 0$, $\Delta^* = \Delta$, *and*

$$\Delta(ab) := \Delta a \, b + (-1)^{\varepsilon\tilde{a}} a \, \Delta b + 2\{a, b\} \tag{53}$$

for all a, b.

Proof. Notice that $\operatorname{grad}(1) = 0$. By (28), we have $\langle \Delta a, b \rangle = -\langle 1, \{a, b\} \rangle$, which implies $\langle \Delta a, b \rangle = (-1)^{\tilde{a}\varepsilon} \langle a, \Delta b \rangle$ due to the symmetry of the bracket. Identity (53) follows by applying (26) to (51). $\square$

It follows that up to a factor of 2, (52) is the unique generating operator for the bracket given by Proposition 2.1. Thus, for algebras with an invariant scalar product we have the uniqueness and existence theorem for generating operators:

Theorem 2.3. *Let A be a commutative associative algebra with unit and an invariant scalar product. For a given bracket in A, the unique generating operator which is self-adjoint and vanishes on constants is, up to a factor of 2, the Laplace operator* div grad.

In particular, in the algebra $\mathfrak{V}(M)$ we have for $\boldsymbol{\psi} = t^w \psi(x)$

$$\operatorname{grad} \boldsymbol{\psi} = t^{w+\lambda}(-1)^{\tilde{a}\tilde{\psi}}\left(S^{ab}\partial_b\boldsymbol{\psi} + \gamma^a w\,\psi\right)\partial_a + t^{w+\lambda+1}\left(\gamma^a\partial_a\boldsymbol{\psi} + \theta w\psi\right)\partial_t. \quad (54)$$

Using the formula for the divergence (23), we after simplification get

$$\Delta\boldsymbol{\psi} = \frac{1}{2}\operatorname{div}\operatorname{grad}\boldsymbol{\psi} = \frac{t^\lambda}{2}\left(S^{ab}\partial_b\partial_a + \left(\partial_b S^{ba}(-1)^{\tilde{b}(\varepsilon+1)} + (2w+\lambda-1)\gamma^a\right)\partial_a + \right.$$
$$\left. w\partial_a\gamma^a(-1)^{\theta(\varepsilon+1)} + w(w+\lambda-1)\theta\right)(t^w\psi)$$

(where we have restored the factor $1/2$), and finally we arrive at formulae (46), (47).

2.4 Properties of the canonical pencil. Examples

Let us study the change of the canonical pencil under a change of γ^a, θ with S^{ab} fixed. Suppose $\bar\gamma^a = \gamma^a + X^a$, $\bar\theta = \theta + \xi$. Let $\bar\Delta_w$ denote the pencil corresponding to $\bar\gamma^a$, $\bar\theta$, and Δ_w denote the pencil corresponding to γ^a, θ. Let Δ and $\bar\Delta$ be the corresponding operators in the algebra $\mathfrak{V}(M)$. Since the canonical pencil depends linearly on the data S^{ab}, γ^a, θ, the difference $\bar\Delta_w - \Delta_w$ is the canonical pencil corresponding to the long bracket given by the matrix

$$\begin{pmatrix} \mathbf{0} & t^{\lambda+1}X^a \\ t^{\lambda+1}X^a & t^{\lambda+2}\xi \end{pmatrix}.$$

Immediately follows that X^a and ξ transform as

$$X^{a'} = J^{-\lambda}X^a\frac{\partial x^{a'}}{\partial x^a}$$
$$\xi' = J^{-\lambda}\left(\xi + 2X^a\partial_a\log J\right).$$

Hence

$$\boldsymbol{X} = X^a\partial_a + \frac{1}{2}\xi\,\hat{w}$$

is a vector field on $\hat M$. Recall that there is a canonical divergence on $\hat M$:

$$\operatorname{div}\boldsymbol{X} = \partial_a X^a(-1)^{\tilde{a}(\varepsilon+1)} + (\lambda-1)\frac{1}{2}\xi. \quad (55)$$

Theorem 2.4. *Under a change of γ^a, θ with S^{ab} fixed we have*

$$\bar\Delta - \Delta = \frac{1}{2}\,\boldsymbol{X}\,(2\hat{w} + \lambda - 1) + \frac{1}{2}\operatorname{div}\boldsymbol{X}\,\hat{w}, \quad (56)$$

or, in terms of pencils:

$$\bar\Delta_w - \Delta_w = \frac{1}{2}(2w+\lambda-1)\left(X^a\partial_a + \frac{1}{2}w\xi\right) + \frac{1}{2}w\operatorname{div}\boldsymbol{X}. \quad (57)$$

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Corollary 2.1. *Decomposing $\boldsymbol{X}$ for $\lambda \neq 1$ we obtain:*

$$\bar{\Delta}_w - \Delta_w = \left(w - \frac{\lambda+1}{2}\right)\mathcal{L}_X^w + \frac{w(w-1)}{\lambda-1}\operatorname{div}\boldsymbol{X} \tag{58}$$

where by

$$\mathcal{L}_X^w := X^a\partial_a - \frac{w}{\lambda-1}\partial_a X^a(-1)^{\tilde{a}(\varepsilon+1)} \tag{59}$$

we denoted the divergence-free part of $\boldsymbol{X}$.

In the case $\lambda = 0$ which is particularly interesting for us, (59) is the usual Lie derivative $\mathcal{L}_X$ of w-densities along the vector field X on M, and we have

$$\bar{\Delta}_w - \Delta_w = \left(w - \frac{1}{2}\right)\mathcal{L}_X - w(w-1)\operatorname{div}\boldsymbol{X} \tag{60}$$

where the canonical divergence of $\boldsymbol{X} = X^a\partial_a + (1/2)\xi\hat{w}$ is expressed as

$$\operatorname{div}\boldsymbol{X} = \partial_a X^a(-1)^{\tilde{a}(\varepsilon+1)} - \frac{1}{2}\xi.$$

In the rest of this subsection we work with operators of weight $\lambda = 0$.

Consider important constructions of canonical pencils.

As follows from (60), the subspace of canonical pencils with $S^{ab} = 0$ is the direct sum of two natural subspaces: $V_1 = \{(2w-1)\mathcal{L}_X \mid X \in \operatorname{Vect}(M)\}$ and $V_2 = \{w(w-1)f \mid f \in C^\infty(M)\}$. The following example provides a subspace which is complementary to $V_1 \oplus V_2$.

Example 2.2 (Canonical pencil associated with a volume form). Fix a basis volume form $\boldsymbol{\rho} = \rho Dx$. For a given bracket of functions specified by a tensor S^{ab}, the following "Laplace–Beltrami type" formula

$$\Delta^{LB}(\psi(Dx)^w) := \frac{1}{2}\rho^{w-1}\partial_a\big(\rho S^{ab}\partial_b(\rho^{-w}\psi)\big)(-1)^{\tilde{a}(\varepsilon+1)}(Dx)^w \tag{61}$$

defines a self-adjoint operator Δ^{LB} on densities, vanishing on the unit. Expanding (61) and comparing with (47), it is easy to see that it gives a canonical pencil with the following γ^a and θ:

$$\gamma^a = -S^{ab}\partial_b\log\rho$$
$$\theta = S^{ab}\partial_b\log\rho\cdot\partial_a\log\rho = \gamma^a\gamma_a.$$

Here the upper connection comes from a genuine connection $\gamma_a := -\partial_a\log\rho$ in $\operatorname{Vol}M$ (which is flat).

It was exactly the pencil Δ_w^{LB} of Example 2.2 that was the main object of study in [14].

Using the canonical pencil Δ_w^{LB} associated with a volume form it is possible to give a convenient parametrization of *all* canonical pencils. If we fix a volume form $\boldsymbol{\rho}$, then every canonical pencil has the appearance

$$\Delta_w = \Delta_w^{LB} + \frac{1}{2}(2w-1)\mathcal{L}_X + w(w-1)f \tag{62}$$

for some vector field X and a scalar function f. Conversely, every pencil (62) is a canonical pencil. This is a decomposition (depending on a choice of $\boldsymbol{\rho}$) of the space of canonical pencils into the direct sum of three subspaces. If Δ_w is given by (62), then

$$\gamma^a = -S^{ab}\partial_b \log \rho + X^a$$
$$\theta = S^{ab}\partial_b \log \rho \, \partial_a \log \rho + 2\partial_a X^a(-1)^{\tilde{a}(\varepsilon+1)} + f.$$

We can apply this to obtain a coordinate-free decomposition of an arbitrary second order linear differential operator acting on densities of fixed weight w_0 on a manifold M.

Any such operator in local coordinates has the appearance

$$L = \frac{1}{2}S^{ab}\,\partial_b\partial_a + T^a\partial_a + R. \tag{63}$$

Clearly, the principal symbol S^{ab} defines a bracket on M; from the subprincipal symbol of L one can construct an upper connection γ^a in $\mathrm{Vol}\,M$ by the formula

$$\gamma^a = \frac{1}{2w_0-1}\left(2T^a - \partial_b S^{ba}(-1)^{\tilde{b}(\varepsilon+1)}\right) \quad \text{if } w_0 \neq \frac{1}{2}. \tag{64}$$

In the case $w_0 = 1/2$ the subprincipal symbol $2T^a - \partial_b S^{ba}(-1)^{\tilde{b}(\varepsilon+1)}$ is a vector field, so it does not give any upper connection.

If we choose a volume form $\boldsymbol{\rho} = \rho Dx$, then we can write L as

$$L = \Delta_{w_0}^{LB} + \mathcal{L}_Q + f \tag{65}$$

where Q is a vector field and f is a scalar function on M, both uniquely defined by L. $\mathcal{L}_Q$ stands for a Lie derivative on w_0-densities. Indeed, let $\Gamma_a = -\partial_a \log \rho$ be the connection in $\mathrm{Vol}\,M$ generated by $\boldsymbol{\rho}$, and $\Gamma^a = S^{ab}\Gamma_b$ be the corresponding upper connection. The vector field Q is defined by the formula

$$Q^a = \frac{1}{2}(2w_0-1)(\gamma^a - \Gamma^a) = \frac{1}{2}\left(2T^a - \partial_b S^{ba}(-1)^{\tilde{b}(\varepsilon+1)} - (2w_0-1)\Gamma^a\right). \tag{66}$$

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(Notice that Q makes sense for all w_0 including $w_0 = 1/2$.) Then one can see that $L - \Delta_{w_0}^{LB} - \mathcal{L}_Q$ is an operator of the zeroth order, i.e. the multiplication by a scalar function, hence we get (65). Formula (65) can be viewed as a replacement of the coordinate description (63) where instead of coefficients depending on a coordinate system we consider a vector field Q and a scalar function f depending on a volume form $\boldsymbol{\rho}$.

What is the relation of an individual operator L with canonical pencils?

We can use decompositions (62) and (65). The general picture is as follows. There is a specialization map $\Delta_w \mapsto \Delta_{w_0} = L$ from the linear space of all canonical pencils Δ_w to the linear space of all second order differential operators on densities of a fixed weight w_0. This map is an isomorphism for all non-singular values $w_0 \neq 0, 1, \frac{1}{2}$:

Theorem 2.5. *Let L be a second order differential operator acting on densities of weight w_0. If $w_0 \neq 0, 1, \frac{1}{2}$, then there exists a unique canonical pencil Δ_w passing through L, i.e. $L = \Delta_{w_0}$. Namely,*

$$\Delta_w = \Delta_w^{LB} + \frac{2w - 1}{2w_0 - 1}\,\mathcal{L}_Q + \frac{w(w - 1)}{w_0(w_0 - 1)}\,f \tag{67}$$

where Q and f are given by (65).

For $w_0 = \frac{1}{2}$ the image of the specialization map consists of all self-adjoint operators (recall that the space of half-densities has a scalar product) and the kernel is the subspace $V_1 = \{(2w - 1)\mathcal{L}_X\}$. For $w_0 = 0$ the image of the specialization map consists of all operators vanishing on constants and the kernel is the subspace $V_2 = \{w(w - 1)f\}$. Similarly we can describe the specialization map for $w_0 = 1$.

Example 2.3 (Canonical pencil associated with a connection). The construction of Example 2.2 can be generalized for an arbitrary connection γ_a in $\mathrm{Vol}\,M$: given a bracket of functions, both γ^a and θ can be defined by the connection as $\gamma^a = S^{ab}\gamma_b$ and $\theta = \gamma^a\gamma_a$. In a more abstract language, the operator Δ_w is (up to 1/2) the Laplace operator div grad on w-densities, where grad is the covariant gradient w.r.t. the induced connection $w\gamma_a$, and $\mathrm{div} = \mathrm{div}_\gamma$ is the divergence of vector fields on M defined by the connection γ_a in $\mathrm{Vol}\,M$. In particular, if a linear connection in TM is given, there is an associated connection in the bundle $\mathrm{Vol}\,M$, namely $\gamma_a = -\Gamma_{ab}^b(-1)^{\tilde{b}}$, in the standard notation, with the curvature given by the trace of the Riemann tensor $-R_{abc}^c(-1)^{\tilde{c}}$. (Notice that this gives an example of a divergence operator with a possibly non-zero curvature.)

Examples 2.2 and 2.3 explain the geometrical meaning of θ.

Consider the restriction of the general formulae for the transformation of the canonical pencil under a change of γ^a, θ to canonical pencils defined by a connection γ_a. A change of connection is given by a covector field X_a:

$$\bar{\gamma}_a = \gamma_a + X_a. \tag{68}$$

Since θ is defined by γ_a as $\gamma^a\gamma_a$, we get for its change

$$\bar{\theta} = \theta + 2\gamma^a X_a + X^a X_a. \tag{69}$$

Proposition 2.3. *For operators defined by a bracket on M and a connection γ_a, the change of the operator under a change of connection is given by the formula*

$$\bar{\Delta} - \Delta = \frac{1}{2}(2w - 1)\mathcal{L}_X - w(w - 1)\left(\mathrm{div}_\gamma X - \frac{1}{2}X^a X_a\right) \tag{70}$$

where $\bar{\gamma}_a = \gamma_a + X_a$, $X^a = S^{ab}X_b$.

It follows that covector fields X considered as functions of two connections: $X_a = X_a(\gamma, \bar{\gamma}) = \bar{\gamma}_a - \gamma_a$, possess the following groupoid property. If for three connections $\gamma, \bar{\gamma}, \bar{\bar{\gamma}}$ with $\bar{\gamma}_a = \gamma_a + X_a, \bar{\bar{\gamma}}_a = \bar{\gamma}_a + Y_a$ the equations

$$\mathrm{div}_\gamma X - \frac{1}{2}X^a X_a = 0 \tag{71}$$

$$\mathrm{div}_{\bar{\gamma}} Y - \frac{1}{2}Y^a Y_a = 0 \tag{72}$$

are satisfied, then the equation

$$\mathrm{div}_\gamma(X + Y) - \frac{1}{2}(X + Y)^a(X + Y)_a = 0 \tag{73}$$

is satisfied for $X + Y$. Equations (71), (72), (73) are generalization of the "Batalin–Vilkovisky equations" of the "master groupoid" discovered in [14]. It follows that the specialization of Δ_w to $w_0 = \frac{1}{2}$ (operators on half-densities) does not depend on a connection γ_a but only on its orbit w.r.t. a groupoid action: $\gamma_a \mapsto \gamma_a + X_a$ where X satisfies (71).

Remark 2.4. A special feature of the connection in Example 2.2 is flatness. Every flat connection in $\mathrm{Vol}\,M$ is locally represented by a 1-form $\gamma_a = -\partial_a \log \rho$. On the intersections we get $\log \rho - \log \rho' = \log J + c$, where c is a local constant. Clearly, c is a 1-cocycle. The local functions ρ can be glued to a nonvanishing volume form if and only if the cohomology class $[c] \in H^1(M; \mathbb{R})$ equals zero. We may say that every flat connection in $\mathrm{Vol}\,M$ comes from a volume form up to the described "twist".

It is tempting to relate flatness of a connection with properties of the operator Δ and the long bracket. This we shall do in Section 3.

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3 Jacobi identity and flatness

In this section we shall focus on the case of odd operators and odd brackets. For them the general analysis performed above can be advanced further. The main tool of classification will become the operator Δ^2. There is a sharp contrast with even operators and the corresponding even brackets, for which no similar development is possible.

3.1 Jacobi identity for a long bracket

In the previous sections, a "bracket" meant just a bilinear concomitant satisfying the derivation property w.r.t. each of its arguments. Now we want to move further and explore the possibility of imposing a Jacobi identity. It turns out, however, that this will be possible only for odd brackets. Indeed, notice that all brackets considered above are symmetric — compared to Lie brackets, which are antisymmetric. What kind of a Jacobi identity can be introduced for a symmetric bracket? The usual Jacobi identity in Lie algebras can be reformulated as either of the following properties: the linear map $a \mapsto [a,\ ]$ takes the bracket to the commutator of operators, or the operator $[a,\ ]$ for each element of the algebra is a derivation of the bracket. The symmetry and linearity conditions for our brackets have the form

$$\{a,b\} = (-1)^{\tilde{a}\tilde{b}}\{b,a\} \tag{74}$$

$$\{ka,b\} = (-1)^{\tilde{k}\varepsilon}k\{a,b\}, \ \{a,bk\} = \{a,b\}k; \tag{75}$$

hence it would make sense to consider a modified bracket $[a,b] := (-1)^{\tilde{a}\varepsilon}\{a,b\}$, for which the same conditions read as

$$[a,b] = (-1)^{\tilde{a}\tilde{b}+\tilde{a}\varepsilon+\tilde{b}\varepsilon}[b,a]$$

$$[ka,b] = k[a,b], \ [a,bk] = [a,b]k.$$

Now, for an even bracket we have $[a,b] = \{a,b\}$, and symmetry: $[a,b] = (-1)^{\tilde{a}\tilde{b}}[b,a]$, still holds. Since the commutator of operators is antisymmetric, $[A,B] = -(-1)^{\tilde{A}\tilde{B}}[B,A]$, there is no hope for $a \mapsto \operatorname{ad}a = [a,\ ]$ to take brackets to brackets. Is it possible, however, to have $\operatorname{ad}a$ as a derivation of the even bracket? Suppose this is satisfied:

$$[a,[b,c]] = [[a,b],c] + (-1)^{\tilde{a}\tilde{b}}[b,[a,c]] \tag{76}$$

for all a,b,c. Rearranging cyclically and adding with suitable signs, we arrive at the following

Proposition 3.1. *If for an even symmetric bracket the "fake Jacobi" property* (76) *is satisfied, then all triple brackets vanish:* $[[a,b],c] = 0$.

In the differential-geometric situation for a bracket $[f,g] = S^{ab}\partial_b f \partial_a g$ with a symmetric even tensor S^{ab}, this implies $S^{ab} \equiv 0$.

Conclusion: there is no way of imposing Jacobi identity for an even symmetric bracket. Hence, we have to concentrate on odd brackets.

From now on all brackets are odd.

For an odd symmetric bracket $\{a,b\}$, we have $[a,b] = (-1)^{\tilde{a}}\{a,b\}$, and the symmetry condition (74) for $\{a,b\}$ becomes the antisymmetry condition for $[a,b]$ with a shift of parity:

$$[a,b] = -(-1)^{(\tilde{a}+1)(\tilde{b}+1)}[b,a]$$

The standard Jacobi condition w.r.t. the shifted parity for $[a,b] = (-1)^{\tilde{a}}\{a,b\}$, translates into the condition

$$\{a,\{b,c\}\} = (-1)^{\tilde{a}+1}\{\{a,b\},c\} + (-1)^{(\tilde{a}+1)(\tilde{b}+1)}\{b,\{a,c\}\}, \qquad (77)$$

or

$$(-1)^{\tilde{a}\tilde{c}}\{\{a,b\},c\} + (-1)^{\tilde{c}\tilde{b}}\{\{c,a\},b\} + (-1)^{\tilde{b}\tilde{a}}\{\{b,c\},a\} = 0 \qquad (78)$$

for the symmetric bracket $\{a,b\}$. Notice that in $\{a,b\}$ the left opening bracket is odd. Historically, when Lie superalgebras first appeared in topology, they were written using a symmetric bracket, and the Jacobi identity appeared for them exactly in the form (78).

In the sequel we continue to work with the symmetric brackets. (Notice the different choice of signs in [14].)

After this algebraic digression, let us return to our geometric situation. Suppose on a manifold M is given an odd symmetric bracket. It is specified by the master Hamiltonian as

$$\{f,g\} = ((S,f),g) = S^{ab}\,\partial_b f \partial_a g (-1)^{\tilde{a}\tilde{f}} \qquad (79)$$

where $S = \frac{1}{2}S^{ab}p_b p_a$ is an odd function on T^*M. Parentheses stand for the canonical Poisson bracket on T^*M. Clearly, see, e.g., [14], the Jacobi identity for (79) is equivalent to the equation

$$(S,S) = 0 \qquad (80)$$

on T^*M. (Notice that for an even symmetric bracket with the even master Hamiltonian this would be an empty condition.)

Let us apply this to a long bracket on M, which is a usual bracket on the manifold $\hat{M}$. Consider for simplicity long brackets of weight 0.

For an odd long bracket specified by a master Hamiltonian $\hat{S}$ by formula (34) (with $\lambda = 0$) we see that it satisfies the Jacobi identity if and only if $\hat{S}$ satisfies the equation $(\hat{S}, \hat{S}) = 0$, with the canonical Poisson bracket on $T^*\hat{M}$.

To get it more explicitly, we can express the canonical bracket on $T^*\hat{M}$ in a "$(D+1)$-formalism", separating variables related to M from the extra variables t, p_t. For Hamiltonians on the extended space $\hat{M}$ we have:

$$(F, G)_{\hat{M}} = (F, G)_M + \frac{\partial F}{\partial p_t}\frac{\partial G}{\partial t} - \frac{\partial F}{\partial t}\frac{\partial G}{\partial p_t},$$

where for clarity we denoted by $(F, G)_{\hat{M}}$ the canonical bracket on $T^*\hat{M}$ and by $(F, G)_M$ the bracket on T^*M with t, p_t considered as parameters. Applying this to

$$\hat{S} = \frac{1}{2}\left(S^{ab}p_b p_a + 2t\gamma^a p_a p_t + t^2\theta p_t^2\right), \tag{81}$$

we obtain the following theorem. Let

$$S = \frac{1}{2}S^{ab}p_b p_a \tag{82}$$

be the master Hamiltonian for the bracket on M, and

$$\gamma = \gamma^a p_a \tag{83}$$

be the local Hamiltonian specifying the upper connection in $\mathrm{Vol}\, M$.

Theorem 3.1. *The Jacobi identity for an odd long bracket of weight zero specified by the "extended" master Hamiltonian* (81) *is equivalent to the following equations:*

$$(S, S) = 0 \tag{84}$$
$$(S, \gamma) = 0 \tag{85}$$
$$(S, \theta) + (\gamma, \gamma) = 0 \tag{86}$$
$$(\gamma, \theta) = 0 \tag{87}$$

Let us analyze the geometric meaning of these equations.

Equation (84) means that the bracket on M satisfies the Jacobi identity, i.e., M is an odd Poisson (Schouten) manifold with the Poisson tensor S. It follows that the operator $D := (S, \ )$ on functions on T^*M is a differential. Hence we can rewrite (85) as

$$D\gamma = 0, \tag{88}$$

and (86) as

$$D\theta + (\gamma, \gamma) = 0. \tag{89}$$

Equation (88) is nothing but the condition of *flatness* for the upper connection γ. Let us explain.

For any upper connection associated with an odd bracket, curvature makes sense if the bracket is a Poisson bracket. Since for "curvature" we need a Lie bracket (a commutator) we cannot discuss curvature for upper connections associated with arbitrary symmetric tensors S^{ab}, for example, when S^{ab} is even.

Remark 3.1. The curvature of a usual ("lower") connection is defined either in terms of the exterior differential of a local connection form or by comparing the commutator of covariant derivatives with the commutator of vector fields. For an upper connection or a "contravariant derivative" ∇^ω, curvature can be introduced in terms of the bracket of 1-forms that comes from a Poisson structure.

Remark 3.2. This bracket of 1-forms is a particular case of the bracket on arbitrary forms known as the "Koszul bracket" for the even Poisson case. Any even Poisson structure induces an odd Koszul bracket of forms such that raising indices by the Poisson tensor takes it to the canonical odd Schouten bracket of multivector fields. The usual exterior differential d is mapped to the "Lichnerowicz operator", i.e. the Schouten bracket with the Poisson tensor. Hence, for an even Poisson bracket, the "curvature form" of any associated upper connection can be defines as a 2-vector field. An analog of this construction can be carried over for odd Poisson brackets. Such bracket induces an even bracket on forms (an analog of the Koszul bracket), such that it is mapped by raising indices by S^{ab} to the canonical Poisson bracket of functions on T^*M. In this case the exterior differential d is mapped to the operator $D = (S, \ )$. In particular, both upper "connection form" and "curvature form" are Hamiltonians linear and quadratic in p_a, respectively.

Example 3.1. Since $D^2 = 0$, the condition of flatness (88) will be identically satisfied if $\gamma = -D \log \rho = -S^{ab} \partial_b \log \rho$ for some ρ. This is exactly the case of the upper connection coming from a volume form. Substituting this into (89) we get $D\theta + (D \log \rho, D \log \rho) = 0$, or $D\big(\theta - (D \log \rho, \log \rho)\big) = 0$, since D is a derivation of the canonical bracket. As the last term is $\{\log \rho, \log \rho\}$, we have $\theta = \{\log \rho, \log \rho\} +$ (Casimir functions). Notice that (87) is then satisfied identically.

There is an important application to the case when the bracket of functions on M is non-degenerate. Then M is an odd symplectic manifold with

the symplectic form

$$\omega = \frac{1}{2}\, dx^a\, dx^b\, \omega_{ba} \tag{90}$$

where (ω_{ab}) is the inverse matrix for (S^{ab}). To the upper connection γ^a corresponds the usual connection $\gamma_a = \omega_{ab}\gamma^b$. The equation (88) for γ^a is equivalent to $d(dx^a \gamma_a) = 0$, i.e., to the flatness of γ_a in the usual sense. Any flat connection in $\mathrm{Vol}\,M$ comes from a volume form up to a twist (see Remark 2.4), so we can rewrite γ_a as $\gamma_a = -\partial_a \log \rho$. We find ourselves in the situation of Example 3.1. In particular, for the "Brans–Dicke field" θ we get $\theta = \{\log \rho, \log \rho\} = \gamma^a \gamma_a$ (if we assume that there are no odd constants). We arrive at the following theorem:

Theorem 3.2 ("Existence of action"). *If S^{ab} is non-degenerate, then the Jacobi identity for the long bracket is equivalent to the following conditions:*
(1) M is symplectic
(2) $\gamma^a = -S^{ab}\partial_b \log \rho$ for some $\rho = e^{\mathcal{A}}$ (a local volume form)
(3) $\theta = \gamma^a \gamma_a$.

The logarithm of a volume form has the physical meaning of "action". Theorem 3.2 tells that if an odd symplectic bracket can be extended to a long bracket of densities satisfying the Jacobi identity, there is an action function $\mathcal{A}$ (at least, local) such that the long bracket comes from the volume form $\boldsymbol{\rho} = e^{\mathcal{A}}\, Dx$ determined by this action.

For an arbitrary long bracket the coefficients S^{ab}, γ^a and θ are independent degrees of freedom. We see that the Jacobi identity eliminates some degrees of freedom, reducing a long odd Poisson bracket in the non-degenerate case to the Poisson bracket of functions plus a volume form (in general, twisted) as an extra piece of data.

The long bracket arising from a volume form $\boldsymbol{\rho}$ is simply

$$\{\boldsymbol{\psi}; \boldsymbol{\chi}\} = \{\boldsymbol{\rho}^{-w_1}\boldsymbol{\psi}, \boldsymbol{\rho}^{-w_2}\boldsymbol{\chi}\}\boldsymbol{\rho}^{w_1+w_2} \tag{91}$$

for densities of weights w_1 and w_2, where at the r.h.s. stands the bracket of functions defined by S^{ab}. Notice that the r.h.s. of (91) is well-defined even for local volume forms with a non-trivial twist (see Remark 2.4).

3.2 Δ^2 and flatness

Let us first discuss a purely algebraic situation.

Let Δ be a second order operator in a commutative associative algebra A with a unit. Consider the operator Δ^2. In general, $\mathrm{ord}\,\Delta^2 \leqslant 4$. If,

however, Δ is odd, then $\Delta^2 = \frac{1}{2}\left[\Delta, \Delta\right]$, hence ord $\Delta^2 \leqslant 3$. In the sequel we consider odd operators. What is the meaning of the conditions ord $\Delta^2 \leqslant k$ (for $k = 2, 1, 0$)?

Proposition 3.2. *The condition* ord $\Delta^2 \leqslant 2$ *is equivalent to the Jacobi identity for the bracket generated by* Δ.

Recall that the formula for the bracket is (for odd Δ)

$$\{a, b\} = \Delta(ab) - (\Delta a)\, b - (-1)^{\tilde{a}} a\, (\Delta b) + \Delta(1)\, ab. \tag{92}$$

By redefining Δ as $\Delta - \Delta(1)$ the last term can be eliminated.

Proposition 3.2 must be evident, since the bracket (92) is nothing but the polarized principal symbol of Δ, and vanishing of $[\Delta, \Delta]$ modulo operators of order $\leqslant 2$ is equivalent to the vanishing of the canonical Poisson bracket of the principal symbol of Δ with itself.

Proposition 3.3. *The condition* ord $\Delta^2 \leqslant 1$ *is equivalent (in addition to the Jacobi identity for the bracket) to the derivation property*

$$\Delta\{a, b\} = \{\Delta a, b\} + (-1)^{\tilde{a}}\{a, \Delta b\}. \tag{93}$$

Notice that if $\Delta(1) = 0$, then ord $\Delta^2 \leqslant 1$ means that Δ^2 is a derivation of the algebra A. Proposition 3.3 then means that Δ^2 is a derivation of the associative multiplication if and only if Δ is a derivation of the bracket. (Then Δ^2 is also a derivation of the bracket.) See [15].

Finally, if ord $\Delta^2 = 0$, this basically means that $\Delta^2 = 0$. All previous properties hold, i.e., Δ generates an odd Poisson bracket for which it is a derivation; and in addition Δ is a differential.

Odd Poisson algebras endowed with an operator Δ generating the bracket and satisfying $\Delta^2 = 0$ have received the name of *Batalin–Vilkovisky algebras*. We see that imposing the conditions ord $\Delta^2 \leqslant k$, $k = 2, 1, 0$, allows to recover the defining identities of the Batalin–Vilkovisky algebras step by step.

Now let us return to the differential-geometric situation. Let A be the algebra of smooth functions on a manifold M. Any odd operator Δ in A of order $\leqslant 2$,

$$\Delta = \frac{1}{2}\, S^{ab}\, \partial_b \partial_a + T^a\, \partial_a, \tag{94}$$

is specified by a quadratic Hamiltonian $S = \frac{1}{2} S^{ab} p_b p_a$ and an associated upper connection $\gamma^a = \partial_b S^{ba} - 2T^a$ in the bundle Vol M (compare (48)). (We have set for convenience $\Delta(1) = 0$.)

Suppose ord $\Delta^2 \leqslant 2$. That means $(S, S) = 0$, and the odd bracket generated by Δ makes M into a Schouten (= odd Poisson) manifold.

By Proposition 3.3, Δ is a derivation of the Schouten bracket if and only if Δ^2 is a vector field (which is, moreover, a Poisson vector field).

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Theorem 3.3. *Let Δ be an arbitrary odd second order operator* (94), *such that $\Delta(1) = 0$. The properties that Δ is a derivation of the corresponding Schouten bracket and that Δ^2 is a Poisson vector field, are equivalent to the flatness of the upper connection:*

$$D\gamma = 0, \tag{95}$$

where $D = (S,\ )$ and $\gamma = \gamma^a p_a$.

Corollary 3.1. *Let the upper connection γ^a come from a usual connection γ_a in $\mathrm{Vol}\,M$ as $\gamma^a = S^{ab}\gamma_b$. Then Δ is a derivation of the Schouten bracket (equivalently, Δ^2 is a Poisson vector field) if and only if γ_a is flat "in the directions of Hamiltonian vector fields", i.e. $S^*(d\gamma_\#) = 0$ where $S^*(dx^a) = S^{ab}p_b$ and $\gamma_\# = \gamma_a\,dx^a$. In particular, Δ is a derivation of the Schouten bracket in the case if γ_a is flat.*

Indeed, $D\gamma = S^*(d\gamma_\#)$, as raising indices by S^{ab} takes d to D, see Remark 3.2.

Remark 3.3. A statement equivalent to Corollary 3.1 was obtained in the important paper [15]. Notice that the class of operators (94) where γ^a comes from a connection in $\mathrm{Vol}\,M$ coincides with the class of operators given by "abstract divergences" (see Remark 2.1). Divergence operators in $\mathrm{Vect}(M)$ and connections in $\mathrm{Vol}\,M$ are equivalent notions. In particular, a connection is flat if the corresponding divergence is flat, i.e., satisfies (30).

Consider now the algebra of densities $\mathfrak{V}(M)$ endowed with the canonical scalar product (20) and an odd operator Δ in it. Suppose $\Delta^* = \Delta$ and $\Delta(1) = 0$. Such operators (in other words, odd canonical pencils) are in $1-1$ correspondence with odd brackets in $\mathfrak{V}(M)$, i.e., odd long brackets on M (Theorem 2.2).

Theorem 3.4. *Let Δ_w be the canonical pencil corresponding to an odd long bracket on M. If the long bracket satisfies the Jacobi identity, then*

$$\Delta_w^2 = \mathcal{L}_X$$

where X is a Poisson vector field on M. Here $\mathcal{L}_X$ stands for the Lie derivative on w-densities.

(Let us emphasize that X is a vector field on M, which is Poisson w.r.t. the odd bracket of functions.)

Indeed, by Proposition 3.2, the Jacobi identity for a long bracket is equivalent to $\mathrm{ord}\,\Delta^2 \leqslant 2$. However, since Δ can be constructed as the Laplace

operator corresponding to the canonical divergence on $\hat{M}$ (see Theorem 2.3), there is a "jump": if ord $\Delta^2 \leqslant 2$, then automatically ord $\Delta^2 \leqslant 1$. This is an algebraic fact following from the flatness of the canonical divergence (30); compare with Corollary 3.1 and the remark after it. Hence $\Delta^2 = \boldsymbol{X}$ is a vector field on $\hat{M}$.

Lemma 3.1. *The vector field* $\boldsymbol{X} = \Delta^2$ *on* $\hat{M}$ *is divergence-free:*

$$\operatorname{div} \boldsymbol{X} = 0, \tag{96}$$

where div *in* (96) *is the canonical divergence* (23).

Not proving this simple lemma (compare Proposition 2.2 in [14]), we conclude, by Theorem 2.1, that $\boldsymbol{X} = \mathcal{L}_X$ for a vector field on M (which must be Poisson), and Theorem 3.4 follows.

Example 3.2. Consider the symplectic case, i.e., when the bracket on M is non-degenerate. Then, by Theorem 3.2, the Jacobi identity for the long bracket implies that γ^a comes from a flat connection $\gamma_a = -\partial_a \log \rho$ in $\operatorname{Vol} M$, and the canonical pencil $\Delta_w = \Delta_w^{LB}$ is the Laplace–Beltrami pencil of Example 2.2. Then $\Delta_w^2 = \mathcal{L}_X$ for the Hamiltonian vector field

$$X = \operatorname{grad} \frac{\Delta_{\mathrm{can}}(\boldsymbol{\rho}^{1/2})}{\boldsymbol{\rho}^{1/2}}, \tag{97}$$

where Δ_{can} stands for the canonical Laplacian on half-densities [10, 11]. Notice that the Hamiltonian in (97) is well-defined even if the volume form exist only locally. A condition $\Delta_w^2 = 0$ will be equivalent to the Batalin–Vilkovisky equation

$$\Delta_{\mathrm{can}}(\boldsymbol{\rho}^{1/2}) = 0 \tag{98}$$

or

$$\Delta_{\mathrm{can}} \left(e^{\mathcal{A}} Dx \right)^{1/2} = 0 \tag{99}$$

if $\boldsymbol{\rho} = e^{\mathcal{A}}$.

4 Generalizations

4.1 Operators and brackets of non-zero weight

The study of operators and brackets of non-zero weight can have a considerable interest.

Let us briefly indicate some of the statements and formulae for $\lambda \neq 0$ which we have omitted in the previous sections.

For a canonical pencil Δ_w the singular points will be $w = 0$, $w = \frac{1-\lambda}{2}$, $w = 1 - \lambda$. In particular, the role of half-densities will be played by densities of weight $\frac{1-\lambda}{2}$ for general λ.

S^{ab} is no longer a tensor field; it is a tensor density of weight λ. Similar is true for γ^a. It is an upper "connection-density". In contrast, γ_a (when it appears) has to be a genuine connection; it will acquire weight by raising indices with the help of S^{ab}.

An interesting example of operators of non-zero weight acting on densities is well known in integrable systems. Namely, consider a one-dimensional manifold M with a coordinate x, and set $\lambda = 2$. By inspection of formulae (38)–(40) we see that the parameters s, γ, θ of the canonical pencil (which we have for convenience multiplied by 2)

$$\Delta_w = s\,\partial^2 + \left(s_x + (2w+1)\gamma\right)\partial + w\,\gamma_x + w(w+1)\,\theta,$$

where $\partial = d/dx$, under a change of a coordinate $y = y(x)$ transform as follows:

$$s' = s \tag{100}$$

$$\gamma' = \frac{y_x\gamma + y_{xx}s}{(y_x)^2} \tag{101}$$

$$\theta' = \frac{(y_x)^2\theta + 2y_xy_{xx}\gamma + (y_{xx})^2s}{(y_x)^4} \tag{102}$$

We used dashes for the parameters in a new coordinate system. As s is invariant, we can set $s = 1$. Then we get

$$\Delta_w = \partial^2 + (2w+1)\,\gamma\,\partial + w\,\gamma_x + w(w+1)\,\theta$$

where $\gamma_x = d\gamma/dx$. In particular, for $w = -\frac{1}{2}$ we obtain a 'Sturm–Liouville' operator

$$L = \Delta_{-\frac{1}{2}} = \partial^2 - \frac{1}{2}\left(\gamma_x + \frac{1}{2}\theta\right) \tag{103}$$

with the potential $U = \frac{1}{2}\left(\gamma_x + \frac{1}{2}\theta\right)$. It maps densities of weight $-\frac{1}{2}$ to densities of weight $\frac{3}{2}$. A known fact about such Sturm–Liouville operators $L = \partial^2 - U$ is that U has a transformation law involving the Schwarz derivative:

$$U' = (y_x)^{-2}\left(U - \frac{1}{2}\,\mathfrak{S}[y(x)]\right),$$

where $\mathfrak{S}[y(x)] = \frac{y_{xxx}}{y_x} - \frac{3}{2}\left(\frac{y_{xx}}{y_x}\right)^2$ (see, e.g., [6]). In our parametrization (103), it follows from the transformation laws (101), (102) for γ and θ, which are

a very special case of the transformation law for the coefficients of a long bracket.

It will be interesting to study the geometrical meaning of this relation further, in particular, to explore possible links of our constructions with projective connections.

Speaking about odd brackets of weight $\lambda \neq 0$, we can notice that in the odd case "Poisson brackets of functions" make no sense separately from the brackets of densities, since the bracket of functions is a density of weight λ, so the Jacobi identity has to involve densities of all weights. Theorem 3.1 is generalized for arbitrary λ as follows:

Theorem 4.1. *The Jacobi identity for an odd long bracket of weight λ specified by the master Hamiltonian* (35) *is equivalent to the following equations:*

$$(S, S) = 2\lambda S\gamma \tag{104}$$
$$(S, \gamma) = \lambda S\theta \tag{105}$$
$$(S, \theta) + (\gamma, \gamma) = \lambda \gamma \theta \tag{106}$$
$$(\gamma, \theta) = 0 \tag{107}$$

In equations (104)–(107) the quantities S, γ and θ are considered as functions on T^*M whose definition depends on coordinates or on a choice of volume form on M. In particular, as for $\lambda \neq 0$ the Hamiltonian S takes values in λ-densities, the canonical bracket (S, S) does not have an invariant meaning. This is indicated by the presence of a non-invariant term $S\gamma$ in the r.h.s. of (104).

It will be interesting to study for $\lambda \neq 0$ the examples of a non-degenerate S^{ab} (a "symplectic structure taking values in densities") and a long bracket coming from a connection or a volume form.

4.2 Operators of higher order

It is tempting to extend the classification of operators and brackets that we have obtained for the operators of order $\leqslant 2$ to operators of higher order. At the moment, we have more questions than answers concerning this case.

First of all, the algebraic framework for brackets generated by an operator Δ is as follows. For simplicity of notation let Δ be even and A be purely even. Of course, the interesting case is that of an odd Δ. Recall that an operator Δ acting in a commutative associative algebra A has order $\leqslant n$ if and only if all $(n + 1)$-fold commutators $[\ldots [[\Delta, a_1], a_2], \ldots, a_{n+1}]$ vanish (where a_i are arbitrary elements of A). Define a sequence of "higher brackets"

corresponding to the operator Δ as k-fold commutators with elements of A applied to 1:

$$\{a\} = [\Delta, a](1) = \Delta a - \Delta(1) \cdot a$$
$$\{a, b\} = [[\Delta, a], b](1) = \Delta(ab) - \Delta(a) \cdot b - a \cdot \Delta(b) + \Delta(1) \cdot ab$$
$$\{a, b, c\} = [[[\Delta, a], b], c](1) = \Delta(abc) - \Delta(ab) \cdot c - \Delta(ac) \cdot b - \Delta(bc) \cdot a$$
$$+ \Delta a \cdot bc + \Delta b \cdot ac + \Delta c \cdot ab - \Delta(1) \cdot abc$$
$$\ldots$$

up to the n-fold bracket, which simply coincides with the n-fold commutator. The brackets higher than the n-th vanish. Notice that all the brackets $\{a_1, \ldots, a_k\}$ are symmetric. Each of them is the obstruction for the previous bracket to be a derivation (w.r.t. each of the arguments). The k-th bracket is an operator of order $n - k + 1$ on each of its arguments. The top n-th bracket $\{a_1, \ldots, a_n\}$ is a multi-derivation. It is exactly the polarized principal symbol of Δ, i.e., the principal symbol considered as a symmetric multilinear function. We can call the constructed sequence of higher brackets the sequence of *polarizations* of the operator Δ.

For example, for an operator of order $\leqslant 2$ we essentially have just one bracket,

$$\{a, b\} = [[\Delta, a], b](1) = \Delta(ab) - \Delta(a) \cdot b - a \cdot \Delta(b) + \Delta(1) \cdot ab,$$

which is a bi-derivation. It was our main object in the previous sections. (The unary bracket $\{a\} = [\Delta, a](1) = \Delta a - \Delta(1) \cdot a$ was used when we redefined Δ so to have $\Delta(1) = 0$.)

Remark 4.1. The definition of the order of an operator in terms of commutators is traced back to Grothendieck. The above sequence of polarizations for Δ essentially coincides with the operations Φ_k considered by Koszul in [16].

For an odd operator Δ of order $\leqslant n$, imposing the conditions $\operatorname{ord} \Delta^2 \leqslant r$, with $r = 2n - 2, 2n - 3, \ldots$, as in Section 3, will lead to an hierarchy of identities involving the associative multiplication, brackets of various orders and Δ. The resulting structure can be loosely called a "homotopy Batalin–Vilkovisky algebra". This should be further explored.

It is instructive to write down the higher brackets explicitly for an operator of order $\leqslant 3$ in a differential-geometric setting. Consider

$$\Delta = \frac{1}{6} S^{abc} \partial_c \partial_b \partial_a + \frac{1}{2} P^{ab} \partial_b \partial_a + T^a \partial_a + R. \tag{108}$$

Let us set $R = 0$ for simplicity. Then $\{f\} = \Delta f$,

$$\{f, g\} = \frac{1}{2} S^{abc} \big(\partial_c f \cdot \partial_b \partial_a g + \partial_c \partial_b f \cdot \partial_a g \big) + P^{ab} \partial_b f \cdot \partial_a g$$

and

$$\{f, g, h\} = S^{abc} \partial_c f \, \partial_b g \, \partial_a h.$$

An analog of the question considered in the previous sections is how to recover Δ from the binary and ternary brackets, $\{f, g\}$ and $\{f, g, h\}$. Clearly, the problem is to identify in geometrical terms extra data that can be constructed from Δ and together with the brackets will allow to recover Δ.

For operators of the second order such information is contained in the subprincipal symbol, which we are able to interpret as an upper connection. However, for the an operator of the third order (108), this will not give a solution, since T^a will not enter $\mathrm{sub}\,\Delta$. One could look for something like iterated "sub-subprincipal" symbols, possibly defined using several volume forms.

With this might be related the following construction for operators of the second order: consider the subprincipal symbol as a vector field depending on a connection and take its divergence w.r.t. the other connection. Notice that for operators defined by a (genuine) connection this is an object depending on three connections. Taking one of them at the midpoint of the segment joining the other two, we arrive at a construction of a scalar function from two genuine connections in $\mathrm{Vol}\,M$ if a second order principal symbol S is given:

$$c(\gamma_0, \gamma_1) = \mathrm{div}_{\frac{\gamma_0 + \gamma_1}{2}} (\gamma_1 - \gamma_0) = \partial_a(\gamma_1^a - \gamma_0^a) - \frac{1}{2} (\gamma_{0a} + \gamma_{1a})(\gamma_1^a - \gamma_0^a) \quad (109)$$

(indices are raised by the tensor S^{ab}). It is exactly the 'Batalin–Vilkovisky' term $\mathrm{div}_\gamma X - \frac{1}{2} X_a X^a$ appearing in the equation (70) for the transformation of the canonical pencil corresponding to a connection under a change of connection. (Notice that a convex combination of connections like $\frac{1}{2} (\gamma_0 + \gamma_1)$ makes good sense.)

The function (109) satisfies the cocycle property

$$c(\gamma_0, \gamma_1) + c(\gamma_1, \gamma_2) = c(\gamma_0, \gamma_2)$$

or

$$\mathrm{div}_{\frac{\gamma_0 + \gamma_1}{2}} (\gamma_1 - \gamma_0) + \mathrm{div}_{\frac{\gamma_1 + \gamma_2}{2}} (\gamma_2 - \gamma_1) = \mathrm{div}_{\frac{\gamma_0 + \gamma_2}{2}} (\gamma_2 - \gamma_0)$$

which is just the groupoid property of the 'Batalin–Vilkovisky equations' (71).

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