Semidensities on Odd Symplectic Supermanifolds

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Abstract: We consider semidensities on a supermanifold E with an odd symplectic structure. We define a new Δ-operator action on semidensities as the proper framework for Batalin-Vilkovisky formalism. We establish relations between semidensities on E and differential forms on Lagrangian surfaces. We apply these results to Batalin-Vilkovisky geometry. Another application is to (1.1)-codimensional surfaces in E. We construct a kind of "pull-back" of semidensities to such surfaces. This operation and the Δ-operator are used for obtaining integral invariants for (1.1)-codimensional surfaces.

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1. Introduction

A density of weight σ is a function on a manifold (supermanifold) subject to the condition that under change of coordinates it is multiplied by the σ-th power of the determinant (Berezinian) of the transformation. A density of weight $\sigma = 1$ is a volume form. (We avoid discussion of orientation here.)

In this paper we study semidensities (densities of weight $\sigma = 1/2$) on a supermanifold provided with an odd symplectic structure (an odd symplectic supermanifold). We introduce a differential operator Δ, which acts on semidensities. Our considerations lead to a straightforward geometrical interpretation of the Batalin-Vilkovisky master equation. On the other hand, we elaborate a new outlook for the invariant semidensity defined on (1.1)-codimensional surfaces embedded in an odd symplectic supermanifold [12] and construct integral invariants for these surfaces.

The concept of an odd symplectic supermanifold and a Δ-operator on it appeared in mathematical physics in the pioneer works of I.A.Batalin and G.A.Vilkovisky [4, 5], where these objects were used for constructing covariant Lagrangian version of the BRST quantization (BV formalism). The geometrical meaning of these objects and interpretation of the BV master equation in its terms were studied in [11, 14, 15] and most notably by A.S.Schwarz in [22].

Let us briefly sketch the results of [11, 14, 15, 22].

If an odd symplectic supermanifold is provided with a volume form $d\mathbf{v}$, then one can consider an operator $\Delta_{d\mathbf{v}}$ such that its action on a function on this supermanifold is equal (up to a coefficient) to the divergence of the Hamiltonian vector field corresponding to this function w.r.t. the volume form $d\mathbf{v}$ [11]. This second order differential operator is not trivial because transformations preserving odd symplectic structure do not preserve any volume form (Liouville theorem does not hold in the case of an odd symplectic structure).

We call coordinates $z^A = \{x^1, \ldots, x^n, \theta_1, \ldots, \theta_n\}$ in an odd symplectic supermanifold Darboux coordinates if in these coordinates the Poisson bracket corresponding to the symplectic structure has the canonical form: $\{x^i, \theta_j\} = \delta^i_j$, $\{x^i, x^j\} = 0$.

Consider a special case, where a volume form in some Darboux coordinates is just the coordinate volume form:

$$d\mathbf{v} = D(x,\theta), \qquad (D(x,\theta) = dx^1 \ldots dx^n d\theta_1 \ldots d\theta_n)\,. \tag{1.1}$$

In the following we shall refer to it as to a particular condition for a volume form. Then in this case the operator $\Delta_{d\mathbf{v}}$ is given by the following explicit formula

$$\Delta_{d\mathbf{v}} = \sum_{i=1}^{n} \frac{\partial^2}{\partial x^i \partial \theta_i}\,,$$

and it obeys the condition

$$\Delta^2_{d\mathbf{v}} = 0\,. \tag{1.2}$$

(See Section 2 for details.)

The concept of an odd symplectic supermanifold provided with a volume form is crucial in the geometrical interpretation of BV formalism.

Let f be an even function on an odd symplectic supermanifold with a coordinate volume form (1.1) in some Darboux coordinates and let $d\mathbf{v}' = f d\mathbf{v}$ be a new volume form on it. In general, for the new volume form $d\mathbf{v}'$ neither condition (1.1) in some Darboux coordinates, nor condition (1.2) are true. The main essence of the geometrical formulation of BV formalism can be shortly expressed in the following two statements [14, 22, 15]:

Statement 1. (see [14, 22, 15])

Consider the following three conditions on the volume form $d\mathbf{v}' = f d\mathbf{v}$ and the corresponding Δ-operator:

$$a) \quad \begin{array}{c}\textit{there exist Darboux coordinates such that the volume form } d\mathbf{v}' = f d\mathbf{v} \\ \textit{has the appearence } (1.1) \textit{ in these coordinates}\,,\end{array} \tag{1.3a}$$

$$b) \quad \begin{array}{c}\Delta_{d\mathbf{v}}\sqrt{f} = 0\,, \\ \textit{(the BV master-equation for the master-action } S = \log\sqrt{f}\,)\,,\end{array} \tag{1.3b}$$

$$c) \quad \Delta^2_{d\mathbf{v}'} = 0\,. \tag{1.3c}$$

The implications

$$a) \Rightarrow b) \Rightarrow c)$$

hold. The conditions a), b), c) *are equivalent under some assumptions (see details below).*

Statement 2. (see [22])

The integrand of the BV partition function is a semidensity $\sqrt{f d\mathbf{v}}$, which is a natural integration object over Lagrangian surfaces in odd symplectic supermanifolds. In the case if condition (1.3b) *is fulfilled, the corresponding integral does not change under small variations of Lagrangian surface (the gauge-independence condition).*

The analysis of these statements in [14, 22, 15] is particularly based on the following geometrical observations.

Let ΠT^*M be the supermanifold associated with the cotangent bundle T^*M for an arbitrary manifold M. (ΠT^*M is obtained by changing the parity of fibers in T^*M.) Functions on ΠT^*M correspond to multivector fields on M. The supermanifold ΠT^*M is provided with a canonical odd symplectic structure. The Schouten bracket of multivector fields on M corresponds to the odd Poisson bracket of functions on ΠT^*M. The manifold

M is a Lagrangian surface in ΠT^*M. If dv is a volume form on M, then the odd symplectic supermanifold ΠT^*M provided with the volume form $d\mathbf{v} = dv^2$ satisfies conditions (1.1) and (1.2). In this case the action of operator $\Delta_{d\mathbf{v}}$ on function on ΠT^*M corresponds to the divergence operator on multivector fields on M. The most profound and detailed analysis of these constructions and their relations with Statements 1 and 2 was performed in the paper [22]. Particularly in this paper some important relations were established between differential forms on M and volume forms in ΠT^*M and it was observed that the square root of an arbitrary volume form in an odd symplectic supermanifold is a natural integration object over arbitrary Lagrangian surfaces in this supermanifold.

In this paper we consider an odd symplectic supermanifold $E = E^{n.n}$. We consider semidensities on E. We define a new operator Δ which acts on semidensities. Our new operator Δ is related with the operator considered above, but it does not require any additional structure on E. We see that semidensities in an odd symplectic supermanifold, not volume forms (densities) are naturally related with differential forms on even Lagrangian surfaces. In particularly the action of a Δ-operator on semidensities corresponds to the action of the exterior differential on differential forms. A detailed analysis of the group of canonical transformations for an arbitrary odd symplectic supermanifold E leads us to establishing relations between the calculus of semidensities on E and a calculus of differential forms on even Lagrangian surface.

Our considerations have the following two applications.

In terms of semidensities the BV master equation (1.3b) gets an invariant formulation and the difference between conditions (1.3a, b, c) can be formulated exactly. (In papers [15] and [22] it was stated that conditions (1.3a), (1.3b) and (1.3c) are equivalent, in spite of the fact that a difference between these conditions implicitly follows from Theorem 5 of the paper [22].) Note also that symmetry transformations in BV formalism considered in the paper [23] receive their proper place in the semidensities framework.

Also we come to a new approach for obtaining invariant densities and the corresponding integral invariants on surfaces embedded in an odd symplectic supermanifold with a volume form. (The problem of constructing integral invariants for an *odd* symplectic structure drastically differs from the corresponding problem for the usual symplectic structure (see in details Section 5)).

The exposition is organized as follows.

In Section 2 we recall the basic definitions of an odd symplectic supermanifold and the properties of the Δ-operator acting on functions and defined when a volume form is chosen. Then we consider semidensities and give an intrinsic definition of the Δ-operator acting on semidensities. Using this operator we formulate the BV master equation in an invariant way.

In Section 3 we analyze these objects in terms of the underlying even geometry considering as the basic example the supermanifold ΠT^*M associated with the cotangent bundle of a usual manifold M. We establish a correspondence between differential forms on M and semidensities on the supermanifold ΠT^*M and analyze the basic formulae of the calculus of differential forms in terms of semidensities. We also come to new algebraic operations on differential forms which naturally appear in terms of semidensities.

In Section 4 we consider even ($(n.0)$-dimensional) Lagrangian surfaces in an odd symplectic supermanifold E and study the correspondence between differential forms on these Lagrangian surfaces and semidensities on E. For any given even Lagrangian surface L this correspondence depends on a symplectomorphism identifying ΠT^*L with E. We prove the existence of an identifying symplectomorphism, study identifying symplectomorphisms and corresponding subgroups of canonical transformations, and investigate in detail to what extent the correspondence between semidensities and differential forms depends on a choice of a Lagrangian surface and identifying symplectomorphism. On the base of these considerations we come to statements that generalize results of the paper [22] and we formulate exactly differences between conditions (1.3a), 1.3b) and (1.3c) in the BV formalism geometry.

In Section 5 we provide a natural interpretation of the odd invariant semidensity on (1.1)-codimensional surfaces that was constructed in [13, 12]. We show that this semidensity can be considered as a kind of "pull-back" of a semidensity in the ambient odd symplectic supermanifold. This leads us to a construction of another semidensity and two densities (integral invariants), even and odd, of rank $k = 4$ on (1.1)-codimensional surfaces. These densities seem to be the simplest (having the lowest rank) non-trivial integral invariants on surfaces in an odd symplectic supermanifold provided with a volume form.

The paper contains also three appendices.

In Appendix 1 we briefly sketch the definition of a supermanifold as a functor from the category of Grassmann algebras to the category of sets, suggested and elaborated by A.S.Schwarz [21] (see also [19]), and which we use throughout this paper. This definition makes possible to use the language of points for supermanifolds. (For basic definitions and constructions of supermathematics see books [6, 19, 25].)

In Appendix 2 we give a simple proof of the Darboux theorem for odd symplectic supermanifolds.

In Appendix 3 we prove a technical result about canonical transformations generated by Hamiltonians.

2. Δ-operator on Semidensities

In this Section we recall the definitions and properties of odd symplectic supermanifold and of the Δ-operator on functions. Then we consider semidensities in odd symplectic supermanifold and define the action of Δ-operator on semidensities. Compared to functions this definition is intrinsic and does not require any additional structures (like volume form).

Let $E^{n.n}$ be $(n.n)$-dimensional supermanifold and $z^A = \{x^1, \dots, x^n,\ \theta_1, \dots, \theta_n\}$, be local coordinates on it ($p(x^i) = 0$, $p(\theta_j) = 1$, where p is a parity).

We say that this supermanifold is odd symplectic supermanifold if it is endowed with an odd symplectic structure, i.e. an odd closed non-degenerate 2-form $\Omega = \Omega_{AB}(z)dz^A dz^B$ ($p(\Omega) = 1$, $d\Omega = 0$) is defined on it [6, 18, 19].

In the same way as in the standard symplectic calculus one can relate to the odd symplectic structure the odd Poisson bracket (Buttin bracket) [8,6, 18, 19]:

$$\{f, g\} = \frac{\partial f}{\partial z^A}(-1)^{p(f)p(z^A)+p(z^A)}\Omega^{AB}\frac{\partial g}{\partial z^B}\,, \tag{2.1}$$

where $\Omega^{AB} = \{z^A, z^B\}$ is the inverse matrix to $\Omega_{AB} : \Omega^{AC}\Omega_{CB} = \delta^A_B$.

Hamiltonian vector field

$$\mathbf{D}_f = \{f, z^A\}\frac{\partial}{\partial z^A}\,, \quad \mathbf{D}_f(g) = \{f, g\}, \quad \Omega(\mathbf{D}_f, \mathbf{D}_g) = -\{f, g\} \tag{2.2}$$

corresponds to every function f.

The condition of the closedness of the form defining symplectic structure implies the Jacoby identity:

$$\{f, \{g, h\}\}(-1)^{(p(f)+1)(p(h)+1)} + \text{cycl. permutations} = 0\,. \tag{2.3}$$

.

Using the analog of the Darboux Theorem [24, 22] (see also Appendix 2) one can consider in a vicinity of an arbitrary point coordinates $z^A = \{x^1, \dots, x^n,\ \theta_1, \dots, \theta_n\}$ such that in these coordinates symplectic structure and the corresponding odd Poisson bracket have locally the canonical expressions

$$\Omega = I_{AB}dz^A dz^B\colon \quad \Omega\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = 0,\, \Omega\left(\frac{\partial}{\partial \theta^i}, \frac{\partial}{\partial \theta^j}\right) = 0,\, \Omega\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial \theta^j}\right) = -\delta_{ij}\,,$$

and respectively

$$\{x^i, x^j\} = 0,\, \{\theta_i, \theta_j\} = 0,\, \{x^i, \theta_j\} = -\{\theta_j, x^i\} = \delta^i_j\,,$$

$$\{f, g\} = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial \theta_i} + (-1)^{p(f)}\frac{\partial f}{\partial \theta_i}\frac{\partial g}{\partial x^i}\right)\,. \tag{2.4}$$

These coordinates are called Darboux coordinates. Transformation of Darboux coordinates to another Darboux coordinates is called canonical transformation of coordinates. Respectively transformation of supermanifold that transforms Darboux coordinates to another Darboux coordinates is called canonical transformation.

We consider also odd symplectic supermanifold provided additionally with a volume form:

$$d\mathbf{v} = \rho(z)Dz = \rho(x,\theta)D(x,\theta)\,, \quad (p(\rho)=0)\,. \tag{2.5}$$

$Dz = D(x,\theta)$ is coordinate volume form ($D(x,\theta) = dx^1 ... dx^n d\theta_1 \dots d\theta_n$). Coordinate volume forms in different coordinates are related by Berezinian (superdeterminant) of coordinate transformation [6]:

$$\frac{D\tilde{z}}{Dz} = \mathrm{Ber}\frac{\partial \tilde{z}}{\partial z}, \quad \text{where } \mathrm{Ber}\begin{pmatrix} I_{00} & I_{01} \\ I_{10} & I_{11} \end{pmatrix} = \frac{\det\left(I_{00} - I_{01}I_{11}^{-1}I_{10}\right)}{\det I_{11}}\,. \tag{2.6}$$

We suppose that the volume form (2.5) is non-degenerate, i.e. for the every point z_0 the number part $m(\rho(z_0))$ of $\rho(z_0)$ is not equal to zero.

In the paper [11] we show that an odd symplectic structure (in fact an odd Poisson bracket structure, which might be degenerate) and a volume form allow to define the Δ-operator (or Batalin-Vilkovisky operator; this is the invariant formulation of the operator introduced in BV-formalism [4]). The construction is as follows. The action of Δ-operator on an arbitrary function in an odd symplectic supermanifold provided with a volume form is equal (up to coefficient) to the divergence w.r.t. volume form (2.5) of the Hamiltonian vector field corresponding to this function. Using (2.2) we come to the formula

$$\Delta_{d\mathbf{v}} f = \frac{1}{2}(-1)^{p(f)}\mathrm{div}_{d\mathbf{v}}\mathbf{D}_f = \frac{1}{2}(-1)^f\left((-1)^{p(\mathbf{D}_f z^A + z^A)}\frac{\partial}{\partial z^A}\{f, z^A\} + D_f^A\frac{\partial \log\rho(z)}{\partial z^A}\right)\,. \tag{2.7}$$

In Darboux coordinates:

$$\Delta_{d\mathbf{v}} f = \Delta_0 f + \frac{1}{2}\{\log\rho, f\}\,, \tag{2.8}$$

where $\rho(z)$ is given by (2.5) and

$$\Delta_0 f = \sum_{i=1}^{n}\frac{\partial^2 f}{\partial x^i \partial\theta_i}\,. \tag{2.9}$$

Δ-operator on functions satisfies the relations [5, 14] :

$$\Delta_{d\mathbf{v}}\{f,g\} = \{\Delta_{d\mathbf{v}} f, g\} + (-1)^{p(f)+1}\{f, \Delta_{d\mathbf{v}} g\}\,,$$

$$\Delta_{d\mathbf{v}}(f\cdot g) = \Delta_{d\mathbf{v}} f\cdot g + (-1)^{p(f)} f\cdot\Delta_{d\mathbf{v}} g + (-1)^{p(f)}\{f,g\}\,. \tag{2.10}$$

Operator Δ_0 in (2.9) is not an invariant operator on functions (i.e. it depends on the choice of Darboux coordinates). It can be considered as $\Delta_{d\mathbf{v}}$ operator for coordinate volume form $D(x,\theta)$ in the chosen Darboux coordinates $z^A = \{x^1,\dots,x^n,\,\theta_1,\dots,\theta_n\}$. If

$\tilde{z}^A = \{\tilde{x}^1, \dots, \tilde{x}^n,\ \tilde{\theta}_1, \dots, \tilde{\theta}_n\}$ are another Darboux coordinates then from (2.8) it follows that

$$\Delta_0 f = \widetilde{\Delta}_0 f + \frac{1}{2}\{\log \mathrm{Ber}\frac{\partial z}{\partial \tilde{z}}, f\}\,, \tag{2.11}$$

where $\widetilde{\Delta}_0$ is operator (2.9) in Darboux coordinates $\tilde{z}^A = \{\tilde{x}^1, \dots, \tilde{x}^n, \tilde{\theta}_1, \dots, \tilde{\theta}_n\}$.

Now we consider semidensities on odd symplectic supermanifold. In local coordinates $z^A = \{x^i, \theta_j\}$ they have the appearance $\mathbf{s} = s(z)\sqrt{Dz} = s(x,\theta)\sqrt{D(x,\theta)}$. Under coordinate transformation $z^A = z^A(\tilde{z})$ the coefficient $s(z)$ is multiplied by the square root of the Berezinian of corresponding transformation: $s(z) \mapsto s(z(\tilde{z}))\mathrm{Ber}^{1/2}(\partial z/\partial \tilde{z})$.

We shall define a new operator, which we denote $\Delta^{\#}$, and which will act on the space of semidensities.

Definition *Let* $\mathbf{s}$ *be a semidensity and* $s(z)\sqrt{Dz}$ *be its local expression in some Darboux coordinates* $z^A = \{x^1, \dots, x^n,\ \theta_1, \dots, \theta_n\}$. *The local expression for the semidensity* $\Delta^{\#}\mathbf{s}$ *in these coordinates is given by the following formula:*

$$\Delta^{\#}\mathbf{s} = (\Delta_0 s(z))\sqrt{Dz} = \sum_{i=1}^{n} \frac{\partial^2 s}{\partial x^i \partial \theta_i}\sqrt{D(x,\theta)}\,. \tag{2.12}$$

The semidensity $\Delta^{\#}\mathbf{s}$ *is an odd (even) if semidensity* $\mathbf{s}$ *is an even (odd) semidensity, thus* $\Delta^{\#}$ *is an odd operator.*

Contrary to the operator $\Delta_{d\mathbf{v}}$ on functions, the operator $\Delta^{\#}$ on semidensities does not need any volume structure.

To prove that $\Delta^{\#}$-operator is well-defined by formula (2.12), one has to check that r.h.s. of (2.12) indeed defines semidensity, i.e. if $\{\tilde{z}^A\} = \{\tilde{x}^1, \dots, \tilde{x}^n, \tilde{\theta}_1, \dots, \tilde{\theta}_n\}$ are another Darboux coordinates then

$$\left(\sum_{i=1}^{n} \frac{\partial^2}{\partial x^i \partial \theta_i} s(z)\right)_{z(\tilde{z})} \cdot \left(\mathrm{Ber}\frac{\partial z(\tilde{z})}{\partial \tilde{z}}\right)^{1/2} = \sum_{i=1}^{n} \frac{\partial^2}{\partial \tilde{x}^i \partial \tilde{\theta}_i}\left(s(z(\tilde{z})) \cdot \mathrm{Ber}^{1/2}\frac{\partial z(\tilde{z})}{\partial \tilde{z}}\right)\,. \tag{2.13}$$

First of all we check this condition for infinitesimal canonical transformations. They are generated by an odd function (Hamiltonian) via the corresponding Hamiltonian vector filed. To an odd Hamiltonian $Q(z)$ corresponds infinitesimal canonical transformation $\tilde{z}^a = z^A + \varepsilon\{Q, z^A\}$ generated by the vector field $\mathbf{D}_Q$ in (2.2). To the action of this transformation on the semidensity $\mathbf{s}$ corresponds differential $\delta_Q(s\sqrt{Dz}) = \Delta_0 Q \cdot s\sqrt{Dz} - \{Q, s\}\sqrt{Dz}$, because $\delta s = -\varepsilon\{Q, s\}$ and $\delta Dz = \varepsilon\delta \mathrm{Ber}(\partial z/\partial \tilde{z})Dz = 2\Delta_0 Q Dz$. Using that $\Delta_0^2 = 0$ and relation (2.10) we come to commutation relations $\Delta_0\delta_Q = \delta_Q\Delta_0$. Thus we come to condition (2.13), for infinitesimal transformations.

To check condition (2.13) for arbitrary canonical transformation we need the following

Lemma 1 1. *Every canonical transformation of Darboux coordinates* $\tilde{z}^A = \mathcal{F}^A(z)$ *can be decomposed into canonical transformations* $\mathcal{F}(z) = \mathcal{F}_s\left(\mathcal{F}_p\left(\mathcal{F}_{\mathrm{adj}}(z)\right)\right)$, *where* *a) canonical transformation* $\tilde z = \mathcal{F}_{\rm adj}(z)$*, has the following form*

$$\begin{cases} \tilde x^i(x,\theta)\big|_{\theta=0} = x^i\,, \\ \tilde\theta_i(x,\theta)\big|_{\theta=0} = 0\,, \end{cases} \qquad (i=1,\ldots,n)\,, \tag{2.14a}$$

we call later this canonical transformation of Darboux coordinates adjusted canonical transformation;

b) canonical transformation $\tilde z = \mathcal{F}_p(z)$ *has the form*

$$\begin{cases} \tilde x^i = x^i(x) \\ \tilde\theta_i = \frac{\partial x^m(\tilde x)}{\partial \tilde x i}\theta_m \end{cases} \qquad (i,j=1,\ldots,n)\,, \tag{2.14b}$$

we call later this canonical transformation of Darboux coordinates, which is generated by transformation $\tilde x^i = x^i(x)$ *"point"-canonical transformation;*

c) canonical transformation $\tilde z = \mathcal{F}_s(z)$ *has the following form*

$$\begin{cases} \tilde x^i = x^i \\ \tilde\theta_i = \theta_i + \Psi_i(x) \end{cases} \quad \text{such that } \frac{\partial\Psi_i(x)}{\partial\, x^j} - \frac{\partial\Psi_j(x)}{\partial x^i} = 0\,, (i=1,\ldots,n)\,, \tag{2.14c}$$

we call later this canonical transformation of Darboux coordinates special canonical transformation.

2. *Berezinian of adjusted canonical transformation* (2.14a) *obeys the condition* $\mathrm{Ber}\frac{\partial\tilde z}{\partial z}\big|_{\theta=0} = 1$*, Berezinian of "point" canonical transformation* (2.14b) *is equal to* $\det^2\frac{\partial\tilde x}{\partial x}$*, and Berezinian of special canonical transformation* (2.14c) *is equal to one.*

In particular, numerical part of Berezinian of arbitrary canonical transformation is positive.

3. *Adjusted canonical and special canonical transformations of Darboux coordinates* (2.14a, 2.14c) *are canonical transformations generated by Hamiltonian, i.e. they can be included in one-parametric family of canonical transformations of Darboux coordinates generated by an odd Hamiltonian:*

$$\exists Q(z,t)\colon \quad \frac{dz_t}{dt} = \{Q, z_t\},\ 0\le t\le 1 \text{ such that } z_0 = z\,, z_1 = \tilde z\,. \tag{2.15}$$

Special canonical transformation (2.14c) *is generated locally by Hamiltonian* $Q = Q(x)$*, such that* $\partial_i Q(x) = \Psi_i(x)$*. There exists unique "time"-independent Hamiltonian* $Q = Q(z)$ *obeying the condition* $Q(x,\theta) = Q^{ik}\theta_i\theta_k + \ldots$*, i.e.* $Q = O(\theta^2)$ *that generates given adjusted canonical transformation.*

Prove this Lemma.

Let $\tilde z = \mathcal{F}(z)$ be arbitrary canonical transformation: $\tilde x^i = f^i(x,\theta) = f^i_0(x) + O(\theta)$ and $\tilde\theta_i = \Psi_i(x) + O(\theta)$. Consider coordinates $\{\bar z^A\} = \{\bar x^i, \bar\theta_i\}$ that are related with coordinates

$\{\tilde{z}^A\}$ by the following special canonical transformation: $\tilde{z}^A = \mathcal{F}_s(\bar{z})$ such that $\tilde{x}^i = \bar{x}^i$ and $\tilde{\theta}_i = \bar{\theta}_i + \Psi_i(g(\bar{x}))$, where $g \circ f_0 = \mathbf{id}$. Then $\bar{x}^i = f^i(x, \theta)$ and $\bar{\theta}_i = O(\theta)$. Now consider coordinates $\{z'^A\} = \{x'^i, \theta'_i\}$ that are related with coordinates $\{\bar{z}^A\}$ by the "point" canonical transformation $\bar{z}^A = \mathcal{F}_p(z')$ generated by functions $\bar{x}^i = f_0^i(x')$. Then it is easy to see that initial coordinates $\{z^A\}$ are related with coordinates $\{z'^A\}$ by adjusted canonical transformation $z'^A = \mathcal{F}_{\mathrm{adj}}(z)$: $x'^i = x^i + O(\theta)$, $\theta'_i = O(\theta)$.

The second statement of Lemma can be proved by easy straightforward calculation of Berezinian (2.6) for transformations (2.14a, 2.14b, 2.14c).

We perform the proof of the statement 3 of Lemma for adjusted canonical transformations in Appendix 3. ∎

Now we return to the proof of relation (2.13).

First we note that from second statement of Lemma it follows that square root operation in (2.13) is well-defined.

From decomposition (2.14) it follows that it is sufficient to check condition (2.13) separately for adjusted, "point", and special canonical transformations. From the third statement of Lemma it follows that for adjusted and special canonical transformations the condition (2.13) can be checked only infinitesimally and this is performed already. For "point" canonical transformation (2.14b) the condition (2.13) can be easily checked straightforwardly using (2.10), (2.11) and the fact that Berezinian of this transformation does not depend on θ. ∎

The action of differential δ_Q corresponding to infinitesimal canonical transformation on semidensities can be rewritten in an explicitly invariant way:

$$\delta_Q \mathbf{s} = Q \cdot \Delta^{\#} \mathbf{s} + \Delta^{\#}(Q\mathbf{s}) = [Q, \Delta^{\#}]_+ \mathbf{s}\,. \tag{2.16}$$

On an odd symplectic supermanifold provided with a volume form $d\mathbf{v}$ (density of the weight $\sigma = 1$) we can construct new invariant objects, expressing them via the semidensity related with volume form and operator $\Delta^{\#}$:

$$\mathbf{s} = \sqrt{d\mathbf{v}} \qquad \text{semidensity } (\sigma = \tfrac{1}{2})\,, \tag{2.17a}$$

$$\Delta^{\#}\mathbf{s} = \Delta^{\#}\sqrt{d\mathbf{v}} \qquad \text{semidensity } (\sigma = \tfrac{1}{2})\,, \tag{2.17b}$$

$$\mathbf{s}\Delta^{\#}\mathbf{s} = \sqrt{d\mathbf{v}}\Delta^{\#}\sqrt{d\mathbf{v}} \qquad \text{density } (\sigma = 1)\,, \tag{2.17c}$$

$$\frac{1}{\mathbf{s}}\Delta^{\#}\mathbf{s} = \frac{1}{\sqrt{d\mathbf{v}}}\Delta^{\#}\sqrt{d\mathbf{v}} \qquad \text{function } (\sigma = 0)\,. \tag{2.17d}$$

From definition (2.12) of $\Delta^{\#}$-operator and relations (2.10) it follows that $\Delta^{\#}$-operator obeys the following properties:

$$(\Delta^{\#})^2 = 0\,,$$

$$\Delta^{\#}(f\cdot\sqrt{d\mathbf{v}}) = (\Delta_{d\mathbf{v}}f)\cdot\sqrt{d\mathbf{v}} + (-1)^f f\cdot\Delta^{\#}\sqrt{d\mathbf{v}}\,, \tag{2.18}$$

and

$$\Delta^2_{d\mathbf{v}}f = \{\frac{1}{\sqrt{d\mathbf{v}}}\Delta^{\#}\sqrt{d\mathbf{v}}, f\}\,. \tag{2.19}$$

We call semidensity $\mathbf{s}$ *closed* semidensity if $\Delta^{\#}\mathbf{s} = 0$ and we call $\mathbf{s}$ an *exact* if there exists another semidensity $\mathbf{r}$ such that $\mathbf{s} = \Delta^{\#}\mathbf{r}$.

In the case if an odd symplectic supermanifold is provided with a volume form $d\mathbf{v}$ such that this volume form is equal to coordinate volume form $D(\tilde{x},\tilde{\theta})$ in some Darboux coordinates $\{\tilde{x}^1,\ldots,\tilde{x}^n,\,\tilde{\theta}_1,\ldots,\tilde{\theta}_n\}$ then evidently $\Delta^{\#}\sqrt{d\mathbf{v}} = 0$. Considering this relation in another Darboux coordinates $z^A = \{x^1,\ldots,x^n,\,\theta_1,\ldots,\theta_n\}$ we come to the formula

$$\Delta_0 \mathrm{Ber}^{1/2}\left(\frac{\partial\tilde{z}}{\partial z}\right) = \sum_{i=1}^n \frac{\partial^2}{\partial x^i\partial\theta_i}\mathrm{Ber}^{1/2}\left(\frac{\partial(\tilde{x},\tilde{\theta})}{\partial(x,\theta)}\right) = 0\,. \tag{2.20}$$

We note that formulae (2.9) and (2.11) for Δ_0 operator were first studied by I.A.Batalin and G.A.Vilkovisky ([4, 5]). In particular they obtained formula (2.20). These results receive its clear geometrical interpretation in terms of semidensities and action of $\Delta^{\#}$-operator on them.

We say that semidensity $\mathbf{s} = s(x,\theta)\sqrt{D(x,\theta)}$ is non-degenerate if a number part $m(s(x,\theta))$ of $s(x,\theta)$ is not equal to zero at any x. Every volume form defines non-degenerate even semidensity by relation (2.17a) and respectively volume form corresponds to every non-degenerate even semidensity. We say that even non-degenerate semidensity $\mathbf{s}$ obeys the BV-master equation if it is closed and we denote by $\mathcal{B}_{\deg}$ a set of these densities.

$$\mathcal{B}_{\deg} = \{\mathbf{s}:\quad \Delta^{\#}\mathbf{s} = 0\,,\quad p(s(x,\theta)) = 0,\quad m(s(x,\theta)) \neq 0\}\,. \tag{2.21}$$

BV-master equation (condition (1.3b)) was not formulated invariantly in [14,22]. Condition $\Delta^{\#}\mathbf{s} = 0$ (closedness of semidensity $\mathbf{s}$) gives invariant formulation to BV master-equation.

3. Differential forms on cotangent bundle and semidensities

We consider in this section basic example of an odd symplectic supermanifold yielded by cotangent bundle of usual manifolds. We clarify geometrical meaning of previous constructions and establish relations between differential forms on manifold and semidensities on this odd symplectic supermanifold.

In the standard sympelctic calculus cotangent bundle of any manifold can be provided with canonical symplectic structure and it can be considered as basic example of symplectic manifold [10]. Basic example of an odd symplectic supermanifold is constructed in the following way. Let M be arbitrary n-dimensional manifold and T^*M be its cotangent bundle. Consider a supermanifold ΠT^*M associated with cotangent bundle T^*M, changing the parity of fibers of cotangent bundle T^*M. Let $\{x^1, \dots, x^n, p_1, \dots, p_n\}$ be canonical coordinates on T^*M corresponding to arbitrary local coordinates $\{x^1, \dots, x^n\}$ on M, i.e. for a form $w \in T^*M$ $p_i(w) = w(\frac{\partial}{\partial x^i})$. Canonical coordinates $z^A = (x^1, \dots, x^n, \theta_1, \dots, \theta_n)$ on ΠT^*M, $(p(\theta_i) = 1)$ correspond to the canonical coordinates $\{x^1, \dots, x^n, p_1, \dots, p_n\}$ on T^*M. Odd coordinates $\{\theta_1, \dots, \theta_n\}$ transform via the differential of corresponding transformation of coordinates $\{x^i\}$ of underlying space M in the same way as coordinates $\{p_1, \dots, p_n\}$ on T^*M:

$$\tilde{x}^i = \tilde{x}^i(x), \quad \tilde{\theta}_i = \sum_{k=1}^{n} \frac{\partial x^k(\tilde{x})}{\partial \tilde{x}^i} \theta_k \,, \quad (i = 1, \dots, n)\,. \tag{3.1}$$

We define canonical odd symplectic structure on ΠT^*M considering these coordinates as Darboux coordinates (2.4). Thus we assign to every atlas $\left[\{x^i_{(\alpha)}\}\right]$ of coordinates on manifold M an atlas $\left[\{x^i_{(\alpha)}, \theta_{j(\alpha)}\}\right]$ of Darboux coordinates on supermanifold ΠT^*M. Pasting formulae (3.1) ensure us that this canonical symplectic structure is well-defined.

Later on we call Darboux coordinates (3.1) on ΠT^*M induced by coordinates on M *Darboux coordinates adjusted to cotangent bundle structure*. Unless otherwise stated we assume further that Darboux coordinates in a supermanifold associated with cotangent bundle are Darboux coordinates adjusted to cotangent bundle structure. (Canonical transformations (3.1), induced by coordinate transformations on the manifold M are "point" canonical transformations (2.14b).)

The relations between the cotangent bundle structure on T^*M and the odd canonical symplectic structure on ΠT^*M reveal in the properties of the following canonical map τ_{M} between multivector fields on M and functions on ΠT^*M:

$$\tau_{M} \left(T^{i_1 \dots i_k} \frac{\partial}{\partial x^{i_1}} \wedge \dots \wedge \frac{\partial}{\partial x^{i_k}} \right) = T^{i_1 \dots i_k} \theta_{i_1} \dots \theta_{i_k}\,. \tag{3.2}$$

This map transforms the Schoutten bracket of multivector fields to the odd canonical Poisson bracket (Buttin bracket) (2.2) of corresponding functions [19,8]:

$$\tau_{M} \left([\mathbf{T_1}, \mathbf{T_2}] \right) = \{ \tau_{M} (\mathbf{T_1}), \tau_{M} (\mathbf{T_2}) \}\,. \tag{3.3}$$

Now we construct a map, that establishes correspondence between differential forms on M and semidensities on ΠT^*M. We consider arbitrary Darboux coordinates $\{x^1, \dots, x^n, \theta_1, \dots, \theta_n\}$ on ΠT^*M adjusted to cotangent bundle structure and define this map in these Darboux coordinates in the following way:

$$\begin{aligned}
\tau^{\#}_{M}(1) &= \theta_1 \dots \theta_n \sqrt{D(x,\theta)}, \\
\tau^{\#}_{M}(dx^i) &= (-1)^{i+1} \theta_1 \dots \widehat{\theta}_i \dots \theta_n \sqrt{D(x,\theta)}\,, \\
\tau^{\#}_{M}(dx^i \wedge dx^j) &= (-1)^{i+j} \theta_1 \dots \widehat{\theta}_i \dots \widehat{\theta}_j \dots \theta_n \sqrt{D(x,\theta)}\,, \ (i<j), \\
&\dots \\
\tau^{\#}_{M}(dx^{i_1} \wedge \dots \wedge dx^{i_k}) &= (-1)^{i_1 + \dots + i_k + k} \theta_1 \dots \widehat{\theta}_{i_1} \dots \widehat{\theta}_{i_k} \dots \theta_n \sqrt{D(x,\theta)}, \ (i_1 < \dots < i_k),
\end{aligned}$$

$$\tau^{\#}_{M}(f(x)w) = f(x)\tau^{\#}_{M}(w)\,, \quad \text{for every function } f(x) \text{ on } M\,, \tag{3.4}$$

where the sign $\widehat{\ }$ means the omitting of corresponding term. For example if M is two-dimensional space, then $\tau^{\#}_{M}(f(x)) = f(x)\theta_1\theta_2\sqrt{D(x,\theta)}$, $\tau^{\#}_{M}(w_1dx^1 + w_2(x)dx^2) = (w_1\theta_2 - w_2(x)\theta_1)\sqrt{D(x,\theta)}$, $\tau^{\#}_{M}(wdx^1 \wedge dx^2) = -w\sqrt{D(x,\theta)}$.

One can rewrite (3.4) in a more compressed way:

$$\tau^{\#}_{M}(w) = \left(\int w(x,\xi)\exp(\theta_i\xi^i)d^n\xi\right)\sqrt{D(x,\theta)}\,, \tag{3.4a}$$

where $w(x,\xi)$ is a function corresponding to differential form w in the supermanifold ΠTM associated to the tangent bundle TM: $w(x,\xi) = w_{i_1\ldots i_k}\xi^{i_1}\ldots\xi^{i_k}$. Odd coordinates $\{\xi^i\}$ of the fibers in ΠTM transform as differentials $\{dx^i\}$: $\tilde{x}^i = \tilde{x}^i(x) \mapsto \tilde{\xi}^i = \frac{\partial\tilde{x}^i}{\partial x^k}\xi^k$. The square of the map (3.4a) $w \to (\tau^{\#}(w))^2$ transforms differential forms on M to density (volume form) on ΠT^*M and this map was considered in [22].

To prove that (3.4) is well-defined for an arbitrary Darboux coordinates adjusted to cotangent bundle structure we note that under arbitrary coordinate transformation (3.1) the integral in r.h.s. of (3.4a) is multiplied on the $\det(\partial\tilde{x}/\partial x)$ and coordinate volume form $\sqrt{D(x,\theta)}$ is divided on the module of this determinant, because for transformation (3.1):

$$\mathrm{Ber}^{1/2}\left(\frac{\partial(\tilde{x},\tilde{\theta})}{\partial(x,\theta)}\right) = \mathrm{Ber}^{1/2}\begin{pmatrix} \frac{\partial\tilde{x}^i(\tilde{x})}{\partial x^k} & \frac{\partial\tilde{x}^r}{\partial x^k}\frac{\partial^2 x^m}{\partial\tilde{x}^r\partial\tilde{x}^i}\theta_m \\ 0 & \frac{\partial x^k}{\partial\tilde{x}^i} \end{pmatrix} = \left|\det\left(\frac{\partial\tilde{x}^i(x)}{\partial x^k}\right)\right|. \tag{3.5}$$

Remark Map (3.4) establishes correspondence only up to a sign factor because r.h.s. of (3.5) is positive for canonical transformation induced by any coordinate transformation of M. In the case if M is orientable manifold considering only Darboux coordinates such that Jacobian of coordinate transformations is positive one comes to globally defined map. Sign factor depends on a orientaion of M.

We say that semidensity $\mathbf{s}$ corresponds to differential form w (to the linear combination of differential forms $\sum w_k$) if $\mathbf{s} = \tau^{\#}_{M}(w) = \tau^{\#}_{M}(\sum w_k)$.

In the end of this section using correspondence between semidensities and differential forms we consider some standard constructions of differential forms calculus in terms of semidensities and two geometrical operations on differential forms, which are naturally arisen in terms of semidensities.

1) From (3.4a) it is easy to see that an action of operator $\xi^i\frac{\partial}{\partial x^i}$ on the function $w(x,\xi)$ corresponds to the action of exterior differential d on differential form and to the action of $\Delta^{\#}$-operator on semidensity, i.e. the action of $\Delta^{\#}$-operator corresponds to the action of exterior differential:

$$\Delta^{\#}\circ\tau^{\#}_{M} = \tau^{\#}_{M}\circ d\,. \tag{3.6}$$

Closed (exact) semidensity corresponds to closed (exact) differential form.

2) If the semidensity $\mathbf{s}$ in ΠT^*M corresponds to volume form (differential top-degree form w on M) and an odd sympelctic supermanifold ΠT^*M is provided with volume form such that it is equal to the square of this semidensity then the action of operator $\Delta_{d\mathbf{v}}$ corresponds to the divergence w.r.t. to the volume form w on M:

$$\Delta_{d\mathbf{v}} \circ \tau_{_M} = \tau_{_M} \circ \mathrm{div}_w \quad \text{if} \quad d\mathbf{v} = \mathbf{s}^2 \text{ and } \mathbf{s} = \tau_M^{\#} w\,. \tag{3.7}$$

(See also [15, 22].)

3) From (3.2) and (3.4) it follows that

$$\tau_{_M}^{\#}(\mathbf{T}\rfloor w) = \tau_{_M}(\mathbf{T}) \cdot \tau_{_M}^{\#}(w)\,, \tag{3.8}$$

where $\mathbf{T}\rfloor w$ is the inner product of multivector field $\mathbf{T}$ with differential form w.

4) The meaning of relation (2.16) in terms of differential forms is following. In the special case if Hamiltonian Q corresponds to vector field $T^i(x)\frac{\partial}{\partial x^i}$ $(Q = T^i(x)\theta_i)$, then this Hamiltonian induces infinitesimal canonical transformation that corresponds to the infinitesimal transformation of M induced by the vector field $T^i(x)\frac{\partial}{\partial x^i}$. From (3.6, 3.8) it follows that in this case the standard formula for Lie derivative of differential forms ($\mathcal{L}_T w = dw\rfloor T + d(w\rfloor T)$) corresponds to relation (2.16). In a general case canonical transformations of ΠT^*M destroy cotangent bundle structure and mix forms of different degrees. For example if we consider the action of Hamiltonian $Q = L\theta_1 \dots \theta_n$ on a semidensity corresponding to form $w = dx^1 \wedge \dots \wedge dx^n$ then we obtain using (2.16) that $\delta w = dL$.

5) If $a = a_i(x)dx^i$ is 1-form on M then one can see that

$$\tau^{\#}(a \wedge w) = a_i \frac{\partial s}{\partial \theta_i}\sqrt{D(x,\theta)}, \quad \text{where} \quad \tau^{\#}(w) = \mathbf{s}\,. \tag{3.9}$$

It is more natural from point of view of semidensities to consider the following relation between 1-forms on M and semidensities in ΠT^*M. Let $a = a_i dx^i$ be an odd-valued one-form on M with coefficients in arbitrary Grassmann algebra Λ (see Appendix 1). For this form and arbitrary semidensity $\mathbf{s} = s(x,\theta)\sqrt{D(x,\theta)}$ consider a new semidensity $\mathbf{s}'$, which we denote by $a \lceil \mathbf{s}$ such that it is given by relation

$$\mathbf{s}' = a \lceil \mathbf{s} = s(x, \theta_i + a_i)\sqrt{D(x,\theta)}\,. \tag{3.10}$$

Respectively if semidensity $\mathbf{s}$ corresponds to differential form $w = \sum w_k$ then we denote by $a \lceil w$ differential form such that semidensity $a \lceil \mathbf{s}$ corresponds to $a \lceil w$. From (3.10) and (3.4a) it follows that

$$a \lceil w = \sum_{p=0}^{k} \frac{1}{p!} \underbrace{a \wedge \dots \wedge a}_{p \quad \text{times}} \wedge w_{k-p}\,, \quad (k = 0, \dots, n)\,. \tag{3.11}$$

Relations (3.10) and (3.11) define an action of abelian supergroup of differential odd valued one-forms on semidensities and differential forms.

6) Consider also the following algebraic operation on differential forms that seems very natural from the point of view of semidensity calculus. Let $w = \sum w_k$ and $w' = \sum w'_k$ be differential forms on M^n such that top-degree forms w_n and w'_n are not equal to zero. Then we consider a new form

$$\tilde{w} = w * w' \colon \quad \tau^{\#}\tilde{w} = \sqrt{\tau^{\#}_M(w_1) \cdot \tau^{\#}_M(w_2)}\,. \tag{3.12}$$

The condition $w_n \neq 0, w'_n \neq 0$ for top-degree forms makes well-defined a square root operation on corresponding semidensities.

4. Semidensities on E and differential forms on even Lagrangian surfaces

In the previous Section we analyzed relations between differential forms on manifold M and semidensities on supermanifold ΠT^*M using Darboux coordinates in ΠT^*M that are adjusted to cotangent bundle structure of T^*M. (Relations (3.4) are not invariant with respect to an arbitrary canonical transformation of Darboux coordinates.)

In this Section we analyze more general situation. We consider relations between semidensities on an arbitrary odd symplectic supermanifold and differential forms on even Lagrangian surfaces in this supermanifold. Then we apply these results for analyzing relations between conditions (1.3a), (1.3b) and (1.3c) for Batalin-Vilkovisky formalism geometry.

Lagrangian surface in $(n.n)$-dimensional odd symplectic supermanifold $E = E^{n.n}$ is $(k.n-k)$-dimensional surface embedded in this supermanifold such that the restriction of symplectic form on it is equal to zero. We call $(n.0)$-dimensional Lagrangian surface *even Lagrangian surface.* For an odd symplectic supermanifold ΠT^*M an initial underlying n-dimensional manifold M can be considered as an even Lagrangian surface embedded in this supermanifold. (Note that in a case if we consider Λ-supermanifolds, underlying manifold is not necessarily Lagrangian surface.)

If L is an even Lagrangian surface in odd sympelctic manifold E and ΠT^*L is supermanifold associated with cotangent bundle of L, then one can consider correspondence between semidensities on E and differential forms on L provided there is an identifying symplectomorphism between supermanifolds ΠT^*L and E:

$$\text{symplectomorphism} \quad \varphi_{_L} \colon \quad \Pi T^*L \to E \text{ and } \quad \varphi_{_L}\big|_L = \mathbf{id}\,. \tag{4.1}$$

In this case pull-back $\varphi^*\mathbf{s}$ of semidensity $\mathbf{s}$ corresponds to differential forms on L via map (3.4):

$$\tau^{\#}_L\,(w_n + w_{n-1} + \ldots + w_1 + w_0) = \varphi^*_{_L}\mathbf{s}\,, \tag{4.2}$$

where w_k is a differential k-form on L.

This correspondence depends on a choice of identifying symplectomorphism (4.1) Thus at first we study properties of identifying symplectomorphisms.

4.1 Identifying symplectomorphisms for even Lagrangian surfaces

In usual symplectic calculus if L is a Lagrangian surface in a symplectic manifold N then there exists symplectomorphism between tubular neighborhoods of L in T^*L and in N that is identical on L [10]. In general case there is no Lagrangian surface L such that T^*L is symplectomorphic to N.

The nilpotency of odd variables leads to the fact that odd symplectic supermanifolds have more simple structure. Particularly, any $(n.0)$-dimensional surface in $(n.n)$-dimensional supermanifold can be expressed locally by equations $\theta_i - \Psi_i(x) = 0, i = 1, \ldots, n$ in any coordinates $\{x^i, \theta_j\}$. Hence for every even Lagrangian surface in odd symplectic supermanifold $E = E^{n.n}$ its underlying n-dimensional manifold $M' = M'^n(L)$ is an open submanifold in underlying manifold M of E. If E' is a corresponding restriction of supermanifold E with underlying manifold M' then one can prove that there exists a symplectomorphism φ that identifies ΠT^*L with E'. We suppose later that M' coincides with M. For example this is a case if M is a closed connected manifold and $M'(L)$ is also closed. We call such Lagrangian surfaces closed.

Proposition 1 *Let L be an arbitrary closed even Lagrangian surface in odd symplectic supermanifold E. Then there exists an identifying symplectomorphism* (4.1) *between* ΠT^*L *and E.*

Prove this Proposition.

An identifying symplectomorphism can be constructed for every even Lagrangian surface in terms of a suitable Darboux coordinates.

Namely, consider arbitrary atlas $\mathcal{A}(E) = \left[\{x^i_{(\alpha)}, \theta_{j(\alpha)}\}\right]$ of Darboux coordinates on a supermanifold E with closed connected underlying manifold M. (Every coordinates $\{x^i_{(\alpha)}, \theta_{j(\alpha)}\}$ of this atlas are defined on superdomain $\hat{U}_\alpha$ with underlying domain U_α. Functions $x^i_{0(\alpha)}$ that are numerical parts of functions x^i_α, define an atlas $\left[\{x^i_{0(\alpha)}\}\right]$ on underlying manifold M.)

We say that Darboux coordinates $\{x^i, \theta_j\}$ in E are adjusted to Lagrangian surface L if $\theta_1 = \ldots = \theta_n = 0$ on L. Respectively we say that an atlas of Darboux coordinates is adjusted to Lagrangian surface L if all coordinates from this atlas are adjusted to this surface.

Suppose that there already exists an identifying symplectomorphism $\varphi_{_L}$ (4.1) for a given closed even Lagrangian surface L. Let $\mathcal{A}(\Pi T^*L) = \left[\{y^i_{(\alpha)}, \eta_{j(\alpha)}\}\right]$ be an atlas of Darboux coordinates in ΠT^*L adjusted to cotangent bundle structure of ΠT^*L (see (3.1)).

Consider an atlas $\mathcal{A}(E) = \left[\{x^i_{(\alpha)}, \theta_{j(\alpha)}\}\right]$ of Darboux coordinates on E defined by relations

$$\varphi^*_L x^i_{(\alpha)} = y^i_{(\alpha)}\,, \quad \varphi^*_L \theta_{j(\alpha)} = \eta_{j(\alpha)}\,. \tag{4.3}$$

This atlas is adjusted to the Lagrangian surface L. Moreover from definition of this atlas and (3.1) it follows that all transition functions $\Psi_{\alpha\beta}$ of $\mathcal{A}(E)$ on superdomains $\hat{U}_{\alpha\beta}$ with underlying domains $U_{\alpha\beta} = U_\alpha \cap U_\beta$ are "point"-like canonical transformations (2.14b). We also call this atlas on E *an atlas adjusted to cotangent bundle structure of Lagrangian surface* L. It is easy to see that arbitrary atlas of Darboux coordinates on E adjusted to cotangent bundle structure of a given Lagrangian surface L defines some identifying symplectomorphism for this Lagrangian surface via relations (4.3). (Darboux coordinates in r.h.s. of (4.3) adjusted to cotangent bundle structure of ΠT^*L are generated by restriction on L of coordinates $\{x^i_{(\alpha)}\}$ in E.) Thus Proposition 1 follows from the following Lemma

Lemma 2. *For arbitrary even Lagrangian surface L in an odd symplectic supermanifold E there exists an atlas of Darboux coordinates in E adjusted to cotangent bundle structure of this surface.*

Prove this Lemma.

Considering in a vicinity of arbitrary point of E arbitrary Darboux coordinates (see for details Appendix 2) we come to some atlas $\left[\{x^i_{(\alpha)}, \theta_{j(\alpha)}\}\right]$ of Darboux coordinates on $E^{n.n}$. If Lagrangian surface L is defined in this atlas by equations $\theta_{i(\alpha)} - \Psi_{i(\alpha)}(x_\alpha) = 0$, then the condition that surface L is Lagrangian implies that $\partial_i \Psi_j - \partial_i \Psi_j = 0$. Hence changing $\theta_{i\alpha} \to \theta_{i(\alpha)} - \Psi_{i(\alpha)}(x_\alpha)$ we come to the atlas $\mathcal{A}_{\rm adj}$ of Darboux coordinates adjusted to the surface L ($\theta_{i(\alpha)}|_L = 0$).

We show that it is possible to change coordinates in every superdomain U_α for atlas $\mathcal{A}_{\rm adj}$ in a way that all transition functions become "point"-like canonical transformations (2.14b). Prove it by induction.

Without loss of generality consider a case, when a number of charts is countable ($\alpha = 1, \ldots, n, \ldots$). Suppose that we already changed coordinates in a required way for first k charts: all transition functions $z_{(\alpha)} = \Psi_{\alpha\beta}(z_{(\beta)})$ are already "point"-like canonical transformations for $\alpha, \beta = 1, \ldots, k$.

Consider Darboux coordinates $\{z^A_{(\alpha)}\} = \{x^i_{(\alpha)}, \theta_{(\alpha)}\}$ on the superdomain $\hat{U}_\alpha$ (with underlying domain U_α) for $\alpha = k+1$. For every $\beta \leq k$ consider transition function (canonical transformation of coordinates) $z^A_{(\beta)} = \Psi_{\beta\alpha}(z_{(\alpha)})$ in superdomain $\hat{U}_{\beta\alpha}$ (with underlying domain $U_{\beta\alpha} = U_\beta \cap U_\alpha$).

All coordinates are adjusted to Lagrangian surface ($\theta_{i(\alpha)}|_L = 0$), hence from statement 1 of Lemma 1 it follows that one can consider in every superdomain $\hat{U}_{\beta\alpha}$ ($\alpha = k+1, \beta \leq k$) new coordinates $\{\tilde{z}^A_{\alpha\beta}\}$ such that $z^A_{(\beta)} = \Psi_{\beta\alpha}(z_{(\alpha)}) = \mathcal{F}_p \circ \mathcal{F}_{\rm adj}(z_{(\alpha)}) = z^A_{(\beta)}(\tilde{z}_{(\beta\alpha)}(z_{(\alpha)}))$ where $z^A_{(\beta)}(\tilde{z}_{(\beta\alpha)})$ is point-like canonical transformation and $\tilde{z}^A_{(\beta\alpha)}(z_{(\alpha)}))$ is adjusted canonical transformation.

To complete the proof of Lemma we have to define in superdomain $\hat{U}_\alpha$ ($\alpha = k+1$) new coordinates $\{\tilde{z}^A_{(\alpha)}\}$ such that restrictions of these coordinates on superdomains $U_{\beta\alpha}$ coincide with coordinates $\{\tilde{z}^A_{(\beta\alpha)}\}$ constructed above. From statement 3 of Lemma 1 it follows that there exist Hamiltonians $Q_{(\beta\alpha)}$ in $\hat{U}_{\beta\alpha}$ that generate adjusted canonical transformation from coordinates z^A_α to coordinates $\tilde{z}^A_{(\beta\alpha)}$ ($\alpha \leq k+1, \beta \leq k$). From inductive hypothesis and uniqueness of these Hamiltonians it follows that $Q_{(\alpha\beta)} = Q_{(\alpha\gamma)}$ in superdomains $\hat{U}_{\alpha\beta\gamma}$. Hence one can consider an odd Hamiltonian obeying the condition $Q = O(\theta^2)$ on a superdomain $\hat{U}_\alpha$ ($\alpha = k+1$) such that restriction of this Hamiltonian on superdomains $\hat{U}_{\beta\alpha}$ is equal to $Q_{(\beta\alpha)}$. This Hamiltonian generates adjusted canonical transformation from coordinates $\{z^A_{(\alpha)}\}$ to a new required Darboux coordinates $\{\tilde{z}^A_{(\alpha)}\}$ on a superdomain $\hat{U}_\alpha$. ∎

Certainly, the identifying symplectomorphism (4.1) for a given closed even Lagrangian surface L is not unique. To study this point consider (infinite-dimensional) supergroup $Can(E)$ of canonical transformations of supermanifold $E^{n.n}$. Every canonical transformation is Λ-point (element) of this supergroup (see Appendix 1). Supergroup $Can(E)$ acts transitively on the superspace of closed even Lagrangian surfaces. Denote by $Can(L)$ stationary subgroup of supergroup $Can(E)$ for L and consider subgroup $Can_{\text{adj}}(L)$ of supergroup $Can(L)$ such that Λ-points of $Can_{\text{adj}}(L)$ are canonical transformations that are identical on the surface L: $Can_{\mathbf{adj}}(L) \ni F \Leftrightarrow F|_L = \mathbf{id}$. It is easy to see that canonical transformations obeying this condition have following appearance in arbitrary Darboux coordinates adjusted to the surface L:

$$\begin{cases} \tilde{x}^i = x^i + f^i(x,\theta), & \text{where } f^i(x,\theta) = O(\theta) \\ \tilde{\theta}_i = \theta_i + g_i(x,\theta), & \text{where } g_i(x,\theta) = O(\theta^2) \end{cases}, \quad \text{if } \theta_i|_L = 0\,. \tag{4.4}$$

Later we call canonical transformations obeying the condition $F|_L = \mathbf{id}$ *canonical transformation adjusted to Lagrangian surface L*. Adjusted canonical transformation (2.14a) corresponds to transformation (4.4) in adjusted coordinates.

Now consider superspace $\Phi(L)$ of identifying symplectomorphisms for given closed even Lagrangian surface L. (Every identifying symplectomorphism $\varphi_{{}_L}$ is Λ-point (element) of this superspace.) Supergroup $Can_{\text{adj}}(L)$ acts free on superspace $\Phi(L)$ of identifying symplectomorphisms: arbitrary two identifying symplectomorphisms $\varphi_{{}_L}$ and $\varphi'_{{}_L}$ differ on canonical transformation adjusted to the surface L:

$$\varphi'_{{}_L} = F \circ \varphi_{{}_L}\,, \quad \text{where } F\Big|_L = \mathbf{id}\,. \tag{4.5}$$

Consider supergroup $Can_0(E)$ that is unity connectivity component of supergroup $Can(E)$, i.e. canonical transformation F belongs to $Can_0(E)$ if it can be included in one-parametric (continuous) family F_t of canonical transformations ($0 \leq t \leq 1$) such that $F_0 = \mathbf{id}$ and $F_1 = F$. Consider also subgroup $Can_H(E)$ of $Can_0(E)$ such that canonical transformation F belongs to $Can_H(E)$ if it can be included in one-parametric family F_t of canonical transformations ($0 \leq t \leq 1$) generated by some Hamiltonian $Q(x,\theta,t)$: $\dot{F}_t = \{Q, F\}$, $F_0 = \mathbf{id}$ and $F_1 = F$. We call canonical transformations belonging to $Can_H(E)$ *canonical transformations generated by Hamiltonian.*

Consider Lie superalgebra $\mathcal{G}_{adj}(L)$ such that Λ-points (elements) of this superalgebra are odd functions on E ("time"-independent Hamiltonians $Q(x,\theta)$) that obey the following condition

$$Q = Q^{ik}(x,\theta)\theta_i\theta_k, \quad \text{i.e.} \quad Q = O(\theta^2) \tag{4.6}$$

in Darboux coordinates adjusted to Lagrangian surface L. (Lie algebra structure is defined via odd Poisson bracket (2.4).)

One can show that superalgebra $\mathcal{G}_{adj}(L)$ corresponds to supergroup $Can_{\mathrm{adj}}(L)$. Indeed it is is easy to see that arbitrary Hamiltonian obeying condition (4.6) generates one-parametric family of canonical transformations $F_t = \mathcal{E}xp\, tQ$ ($0 \leq t \leq 1$) adjusted to the surface L and $\mathcal{E}xp\, tQ_1 \neq \mathcal{E}xp\, tQ_2$ if $Q_1 \neq Q_2$. (see for detailes Appendix 3). Thus the map $\mathcal{E}xp\, Q$: $\mathcal{G}_{\mathrm{adj}}(L) \to Can_{\mathrm{adj}}(L)$ is well-defined injection. Moreover this exponential map is bijective map. To find the Hamiltonian $Q \in \mathcal{G}_{\mathrm{adj}}(L)$ that generates a given transformation $F \in Can_{\mathrm{adj}}(L)$ ($F = \mathcal{E}xp\, Q$) consider the transformation F in arbitrary atlas $\mathcal{A}$ of Darboux coordinates adjusted to cotangent bundle structure of the surface L. In every coordinates from this atlas transformation F has appearance (4.4), hence according to the statement 3 of Lemma 1 in every coordinates from atlas $\mathcal{A}$ there exists unique "time"-independent Hamiltonian $Q_{(\alpha)}$ obeying condition (4.6) such that this Hamiltonian generates locally this transformation ($Q_{(a)} = -\theta_i f^i(x,\theta) + O(\theta^3)$). One can see that local Hamiltonians $\{Q_{(\alpha)}\}$ do not depend on a choice of coordinates from this atlas. Hence they define uniquely a global Hamiltonian Q in superalgebra $\mathcal{G}_{\mathrm{adj}}(L)$. We come to

Proposition 2

For a given closed even Lagrangian surface L in E arbitrary two identifying symplectomorphisms are related with each other by canonical transformation adjusted to Lagrangian surface. This canonical transformation is generated by "time"-independent Hamiltonian that is defined uniquely by condition (4.6). *In other words supergroup $Can_{\mathrm{adj}}(L)$ acts free on superspace $\Phi(L)$ of identifying symplectomorphisms. The exponential map $\mathcal{E}xp$ from Lie superlalgebra Lie $\mathcal{G}_{\mathrm{adj}}(L)$ to $Can_{\mathrm{adj}}(L)$ is bijection.*

For later considerations we need to study difference between supergroups $Can_0(E)$ (unity connectivity component in $Can(E)$) and supergroup $Can_H(E)$ of canonical transformations generated by Hamiltonian. For this purpose we consider decomposition of group $Can(E)$ of all canonical transformations on subgroups that are isomorphic to $Can_{\mathrm{adj}}(L)$, supergroup $Diff(L)$ of diffeomorphisms of Lagrangian surface L, and supergroup that acts free on the superspace of all even Lagrangian surfaces.

To describe this latter supergroup consider abelian supergroup $\Pi Z^1(L)$ of closed differential one-forms on L, where $Z^1(L)$ is superspace of closed differential forms on L and Π is parity reversing functor. (Λ-points of supergroup $\Pi Z^1(L)$ ($Z^1(L)$) are closed one-forms with odd (even) coefficients from Grassmann algebra Λ.) Supergroup $\Pi Z^1(L)$ is subgroup of abelian supergroup of odd-valued differential one-forms considered in Section 3 (see (3.10)). Superspace $\Pi Z^1(L)$ can be identified with superspace of even closed Lagrangian surfaces in ΠT^*L, because every odd valued differential one-form $\Psi_i dx^i$ can be identified with $(n.0)$-dimensional surface embedded in ΠT^*L given by equations $\theta_i - \Psi_i(x) = 0$. Under this identification closed even Lagrangian surfaces in ΠT^*L correspond to closed forms. There is a natural monomorphism of supergroup $\Pi Z^1(L)$ in supergroup $Can(\Pi T^*(L))$ of *all* canonical transformations of supermanifold ΠT^*L: the special canonical transformation (2.14c) $x^i_{(\alpha)} \to x^i_{(\alpha)}$, $\theta_{j(\alpha)} \to \theta_{j(\alpha)} + \Psi_{i(\alpha)}(x_{(\alpha)})$ corresponds to an element $\Psi_i(x)dx^i$ of supergroup $\Pi Z^1(L)$ in an atlas of Darboux coordinates on ΠT^*L adjusted to cotangent bundle structure of L. Abelian supergroup $\Pi Z^1(L)$ acts free on the superspace of closed even Lagrangian surfaces in ΠT^*L. The action of this supergroup on semidensities in ΠT^*L and arbitrary differential forms on L is defined by operation (3.10).
There is also natural monomorphism of supergroup $Diff(L)$ in supergroup $Can(\Pi T^*(L))$ of *all* canonical transformations of supermanifold ΠT^*L corresponding to point-canonical transformation (see (2.14b) and (3.1)).

Now for supergroups $Diff(L)$ and $\Pi Z^1(L)$ we consider affine supergroup $\Pi Z^1(L) \ltimes Diff(L)$ such that semidirect product is induced by action of diffeomorphisms of L on forms: $[\Psi_1, f_1] \circ [\Psi_2, f_2] = [\Psi_1 + (f_1^{-1})^*\Psi_2, f_1 \circ f_2]$, where $\Psi_1, \Psi_2 \in \Pi Z^1(L)$ are closed odd valued one-forms and $f_1, f_2 \in Diff(L)$ are diffeomorphisms of L. Monomorphisms of supergroups $\Pi Z^1(L)$ and $Diff(L)$ in supergroup $Can(\Pi T^*L)$ considered above define monomorphism ι of the affine supergroup $\Pi Z^1(L) \ltimes Diff(L)$ in the supergroup $Can(\Pi T^*L)$. Thus every identifying symplectomorphism φ_L defines monomorphism $\iota_{\varphi_L} = \varphi_L \circ \iota \circ \varphi_L^{-1}$ of the supergroup $\Pi Z^1(L) \ltimes Diff(L)$ in the supergroup $Can(E)$.

On the other hand consider for arbitrary canonical transformation $F \in Can(E)$ Lagrangian surface $\widetilde{L} = \varphi_L^{-1} \circ F(L)$ in ΠT^*L and closed odd valued one-form Ψ on L corresponding to the Lagrangian surface $\widetilde{L}$. Then canonical transformation $F' = \iota_{\varphi_L}([-\Psi, \mathbf{id}]) \circ F$ of supermanifold E belongs to supergroup $Can(L)$ of canonical transformations that transform Lagrangian surface L to itself. The restriction of the canonical transformation F' on L defines diffeomorphism $f = F'|_L \in Diff(L)$. Thus we define projection map

$$p_{\varphi_L}: \quad Can(E) \to \Pi Z^1(L) \ltimes Diff(L) \tag{4.7}$$

that depends on identifying symplectomorphism.

This map projects subgroup $Can_{\rm adj}(L)$ of canonical transformations adjusted to the surface L to unity element and obeys the condition $p_{\varphi_L} \circ \iota_{\varphi_L} = \mathbf{id}$. We come to the following result: for a given Lagrangian surface L and identifying symplectomorphism $\varphi_{_L}$ arbitrary canonical transformation $F \in Can(E)$ can be decomposed uniquely in the following way:

$$F = \iota_{\varphi_L}([\Psi, f]) \circ F_{\mathrm{adj}} = F_s \circ F_p \circ F_{\mathrm{adj}}\,, \quad \text{where}$$
$$[\Psi, f] = p_{\varphi_L}(F)\,,\ F_s = \iota_{\varphi_L}([\Psi, \mathbf{id}])\,,\ F_p = \iota_{\varphi_L}([0, f])\,,\ F_{\mathrm{adj}} \in Can_{\mathrm{adj}}(L)\,. \tag{4.8}$$

(The decomposition (2.14) in Lemma 1 corresponds to this decomposition.)

One can check the following property of projection map (4.7): If $\varphi_{_L}$, $\varphi'_{_L}$ are two arbitrary identifying symplectomorphisms for a given Lagrangian surface L and $p_{\varphi_L}(F) = [\Psi, f]$, $p_{\varphi'_L}(F) = [\Psi', f']$ then

$$\Psi' - \Psi = d\Phi\,, \quad f' = f_0 \circ f\,, \tag{4.9}$$

where f_0 is diffeomorphism that can be included in one-parametric continuous family f_t of diffeomorphisms such that $f_1 = f$ and $f_0 = \mathbf{id}$. In other words f_0 belongs to group $Diff_0(L)$ that is unity connectivity component of group $Diff(L)$.

According to Proposition 2 conditions (4.9) have to be checked only for infinitesimal canonical transformations (4.4) adjusted to Lagrangian surface L and generated by Hamiltonian (4.6). This can be done by easy straightforward calculations.

From (4.9) it follows that for a given Lagrangian surface L projection map (4.7) defines a map

$$p_L\colon \quad Can(E) \to \Pi H^1(L) \ltimes \pi_0(Diff(L))\,, \tag{4.10a}$$

where $\Pi H^1(L)$ is abelian group of cohomology classes of one-forms on L (with reversed parity) and $\pi_0(Diff(L)) = Diff(L)/Diff_0(L)$ is discrete group of connectivity components of $Diff(L)$. Using decomposition (4.8), relations (4.9) and the fact that supergroup Can_{adj} is normal subgroup in $Can(L)$ one can show that (4.10a) is epimorphism. (Projection map (4.7) is not epimorphism, because supergroup Can_{adj} is not normal subgroup in $Can(E)$.)

One can consider also a composition of epimorphism (4.10a) with natural epimorphism of $\Pi H^1(L) \ltimes \pi_0(Diff(L))$ on $\pi_0(Diff(L))$:

$$\hat{p}_L\colon \quad Can(E) \xrightarrow{p_L} \Pi H^1(L) \ltimes \pi_0(Diff(L)) \to \pi_0(Diff(L))\,. \tag{4.10b}$$

Epimorphisms (4.10a) and (4.10b) allow to check difference between supergroups $Can_0(E)$ and $Can_H(E)$ because

$$\mathbf{ker}\, p_L = Can_H(E) \quad \text{and} \quad \mathbf{ker}\, \hat{p}_L = Can_0(E)\,. \tag{4.11}$$

Namely consider arbitrary canonical transformation F that belongs to the kernel of epimorphism (4.10a). Then for projection map (4.7) $p_{\varphi_L}(F) = [\Psi, f]$ where $\Psi = d\Phi$ and

$f \in Diff_0(L)$. Consider decomposition (4.8) for this canonical transformation F. Then canonical transformation $F_s = \rho([\Psi, \mathbf{id}])$ is generated by Hamiltonian $Q = \Phi(x)$. Canonical transformation $F_s = \rho([0, f])$ is generated by Hamiltonian $Q = K^i(t,x)\theta_i$ where "time"-dependent vector field $K^i(t,x)$ is equal to $f_t^{-1} \circ \dot{f}_t$ for a family f_t of diffeomorphisms that connects diffeomorphism f with identity diffeomorphism. Canonical transformation $F_{\rm adj}$ is generated by some Hamiltonian $Q(x,\theta)$ according to Proposition 2. Hence the kernel of epimorphism (4.10a) belongs to $Can_H(E)$. To prove the converse implication consider one-parametric family F_t of canonical transformations generated by arbitrary Hamiltonian $Q(x,t)$. Decompose for every t transformation F_t by formula (4.8) for arbitrary identifying symplectomorphism φ_L: $F_t = F_s(t) \circ F_p(t) \circ F_{\rm adj}(t)$. Transformations $F_p(t)$ and $F_{\rm adj}(t)$ are generated by Hamiltonians hence transformation $F_s(t)$ is generated by Hamiltonian Q' also. Hence $\Psi = dQ'$,where $p_{\varphi_L} F_p(t) = [\Psi_t, 1]$ and $p_L(F) = [0,1]$ in $\Pi H^1(L) \ltimes \pi_0(Diff(L))$.

The proof of the second relation in (4.11) is analogous. We come to

Proposition 3 *Let L be a closed even Lagrangian surface in an odd symplectic supermanifold E. Let $Can_0(E)$ be unity connectivity component of supergroup $Can(E)$ of canonical transformations of E and $Can_H(E)$ be supergroup of canonical transformations generated by Hamiltonian. Then the following relations between supergroups $Can(E)$, $Can_0(E)$ and $Can_H(E)$ are obeyed:*

$$\begin{gathered} Can(E)/Can_0(E) = \pi_0(Diff(L)), \\ Can(E)/Can_H(E) = \Pi H^1(L) \ltimes \pi_0(Diff(L)), \\ Can_0(E)/Can_H(E) = \Pi H^1(L)\,. \end{gathered}$$

In particularly supergroup $Can_0(E)$ is equal to supergroup $Can_H(E)$ if $H^1(L) = 0$.

Groups $Diff(L)$, $\pi_0(Diff(L))$, $\Pi Z^1(L)$ and $\Pi H(L)$ are isomorphic to groups $Diff(M)$, $\pi_0(Diff(M))$, $\Pi Z^1(M)$ and $\Pi H(M)$ respectively, where M is underlying supermanifold, but isomorphisms are not canonical.

4.2 Relation between semidensities and differential forms on a Lagrangian surface

Now we return to relation (4.2) between semidensities on odd symplectic supermanifold $E = E^{n.n}$ and differential forms on even Lagrangian surfaces. We assume that underlying manifold is orientable (see Remark after (3.5)) and its orientation is fixed. This fixes orientation on even Lagrangian surfaces.

We note also that if we consider the points of supermanifold as Λ-points where Λ is an arbitrary Grassmann algebra, then one have to consider differential forms with coefficients in this algebra Λ (see Appendix 2). It follows from (3.4) that if $\mathbf{s}$ is even (odd) semidensity then k-form in l.h.s. of relation (4.2) has coefficients in Grassmann algebra Λ with parity $p = (-1)^{n-k}$ ($p = (-1)^{n-k+1}$). More precisely denote by S a superspace of semidensities in E. Λ-points of superspace S are even semidensities, i.e. semidensities

$\mathbf{s} = s(x,\theta)\sqrt{D(x,\theta)}$, such that $s(x,\theta)$ are even functions with coefficients in Grassmann algebra Λ. Denote by Ω^k superspace of differential k-forms on even Lagrangian surface L and consider also superspace $\Pi\Omega^k$, where Π is parity reversing functor. Λ-points of superspace Ω^k are differential k-forms with even coefficients from Grassmann algebra Λ, Λ-points of superspace $\Pi\Omega^k$ are differential k-forms, with odd coefficients from Grassmann algebra Λ. Consider a superspace

$$\Omega^*(L) = \Omega^n \oplus \Pi\Omega^{n-1} \oplus \Omega^{n-2} \oplus \Pi\Omega^{n-3} \oplus \Omega^{n-4} \ldots \tag{4.12}$$

Relation (4.2) defines a map:

$$w(L, \varphi_{_L}, \mathbf{s}) = (\tau_L^{\#})^{-1}\varphi_{_L}^*\mathbf{s} \tag{4.13}$$

between superspace S and superspace $\Omega^*(L)$. (Here and later where it will not lead to confusion we denote by w a linear combination of differential forms $w_n + w_{n-1} + \ldots + w_0$.) At what extent map (4.13) depends on a choice of identifying symplectomorphism and on a choice of even Lagrangian surface?

If F is arbitrary canonical transformation of E and $\varphi'_{_L} = F \circ \varphi_{_L}$ then for map (4.13)

$$w(L, \varphi'_{_L}, \mathbf{s}) = w(L, F \circ \varphi_{_L}, \mathbf{s}) = w(L, \varphi_{_L}, F^*\mathbf{s})\,. \tag{4.14}$$

Thus bearing in mind Proposition 2 we study the action of supergroup $Can_{\mathrm{adj}}(L)$ of canonical transformations on semidensities.

Proposition 4

a)*Let* $\mathbf{s}$ *be arbitrary semidensity on odd symplectic supermanifold* $E = E^{n.n}$ *with closed connected underlying manifold* M^n *and* F *be arbitrary canonical transformation of* $E^{n.n}$ *adjusted to a given even Lagrangian surface* L *in* E ($F\big|_L = \mathbf{id}$, *i.e.*$F \in Can_{\mathrm{adj}}(L)$)*.*

Then $(F^*\mathbf{s} - \mathbf{s})\big|_L = 0$.

b) *Arbitrary canonical transformation* F *generated by Hamiltonian* ($F \in Can_H(E)$) *changes arbitrary closed semidensity on an exact form: if* $\Delta^{\#}\mathbf{s} = 0$ *then* $F^*\mathbf{s} - \mathbf{s} = \Delta^{\#}\mathbf{r}$ *. In the case if this transformation is adjusted to Lagrangian surface* L ($F \in Can_{\mathrm{adj}}(L) \subseteq Can_H(L)$)*, then condition* $(F^*\mathbf{s} - \mathbf{s})\big|_L = 0$ *is obeyed also.*

c) *If* $\mathbf{s}$ *and* $\mathbf{s}_1$ *are arbitrary even closed non-degenerate semidensities (* $\mathbf{s}, \mathbf{s}_1 \in \mathcal{B}_{\mathrm{deg}}$ *(see* (2.21))*, differ on an exact semidensity:* $\mathbf{s}_1 - \mathbf{s} = \Delta^{\#}\mathbf{r}$*, and for even Lagrangian surface* L *condition* $(\mathbf{s}_1 - \mathbf{s})\big|_L = 0$ *is obeyed, then there exists a canonical transformation* F *adjusted to* L *such that* $\mathbf{s} = F^*\mathbf{s}_1$.

(We say that semidensity $\mathbf{s}$ is equal to zero on even Lagrangian surface L ($\mathbf{s}\big|_L = 0$) if in Darboux coordinates adjusted to L $\mathbf{s} = s(x,\theta)\sqrt{D(x,\theta)}$ with $s(x,\theta)|_{\theta=0} = 0$.)

The statement a) follows from explicit expression (4.4) for transformation F adjusted to Lagrangian surface L. According to Proposition 2 statement b) have to be checked only for infinitesimal transformations generated by Hamiltonian. For these transformations this statement follows from formula (2.16).

To prove statement c) we consider a following "time"-depending Hamiltonian:

$$Q(t) = \frac{-\mathbf{r}}{\mathbf{s} + t\Delta^{\#}\mathbf{r}}\,, \quad 0 \le t \le 1 \tag{4.15}$$

for any one-parameter family $\mathbf{s}_t = \mathbf{s}_0 + t\Delta^{\#}\mathbf{r}$, $0 \le t \le 1$ of even closed non-degenerate semidensities ($\mathbf{s}_t \in \mathcal{B}_{\rm deg}$ at any t).

It is easy to check that during a "time" t this Hamiltonian generates canonical transformation F_t that transforms $\mathbf{s}_t$ to $\mathbf{s}$ ($F_t^*\mathbf{s}_t = \mathbf{s}$). Indeed according to (4.15) and (2.16) if transformation F_t obeys the conditions $\dot{F}_t = \{Q, F_t\}$ and $F_0 = \mathbf{id}$ then

$$\frac{d}{dt} F_t^* \mathbf{s_t} = F_t^* \Delta^{\#}\mathbf{r} + F_t^* \left(\Delta^{\#}\left(Q(t)\mathbf{s}_t\right)\right) = 0 \Rightarrow F_t^*\mathbf{s}_t = \mathbf{s}_0\,.$$

Consider Hamiltonian (4.15) for semidensities $\mathbf{s}, \mathbf{s}_1 \in \mathcal{B}_{\rm deg}$ with $\Delta^{\#}\mathbf{r} = \mathbf{s}_1 - \mathbf{s}$ choosing $\mathbf{r}$ in such a way that $\mathbf{r} = O(\theta^2)$ in coordinates adjusted to L. Then Hamiltonian (4.15) leads to canonical transformation F_t that is adjusted to L at any t and the transformation $F = F_1$ transforms $\mathbf{s}_1$ to $\mathbf{s}$. ∎

Now we use this Proposition for analyzing relation (4.2) for a given even ($(n.0)$-dimensional) Lagrangian surface L.

1. According to Proposition 2 two identifying symplectomorphisms for a given even Lagrangian surface differ on canonical transformation adjusted to this surface. Hence from statement a) of Proposition 4 and condition (4.14) for map (4.13) it follows that top-degree form w_n in (4.13) does not depend on a choice of identifying symplectomorphism $\varphi_{_L}$: for a given even Lagrangian surface L relations (4.13) induce a well-defined map

$$V(L, \mathbf{s}) = w_n(L, \mathbf{s}) \stackrel{\rm def}{=} w_n(L, \varphi_{_L}, \mathbf{s})\,, \tag{4.16}$$

where $\varphi_{_L}$ is arbitrary identifying symplectomorphism $\varphi_{_L}$ for Lagrangian surface L.

Formula (4.16) defines the map from superspace S of semidensities in $E = E^{n.n}$ to the superspace $\Omega^n(L)$ of top-degree forms on L. This means that semidensity can be considered as well-defined integration object over even Lagrangian surface. This corresponds to general result that semidensity can be considered as an integration object over arbitrary $(n-k.k)$-dimensional Lagrangian surface. (See [22] for corresponding construction and [15], [3] for explicit formulae.)

2. Consider restriction of the map (4.13) on the superspace $\mathcal{B}$ of closed semidensities. If semidensity $\mathbf{s}$ is closed then it follows from statement b) of Proposition 4 and relation

(3.6) that for a given even Lagrangian surface L under changing of identifying symplectomorphism $\varphi_{_{L,E}} \mapsto \varphi'_{_{L,E}} = F_{\rm adj} \circ \varphi_{_{L,E}}$, corresponding differential forms in (4.13) change on exact forms: if $\Delta^{\#}\mathbf{s} = 0$ then

$$w_k(L, \varphi'_{_L}, \mathbf{s}) = w_k(L, F \circ \varphi_{_L}, \mathbf{s}) = w_k(L, \varphi_{_L}, F^*\mathbf{s}) = w_k(\varphi_{_L}, \mathbf{s}) + dw_{k-1}(L, \varphi_{_L}, \mathbf{r})\,. \quad (4.17)$$

In particular w_0 (constant) as well as w_n do not depend on identifying symplectomorphism.

Projection of superspaces Ω^k in (4.12) on a superspaces H^k of cohomology classes for $k \leq n-1$ induces projection of superspace Ω^* on superspace:

$$\Omega^n(L) \oplus \Pi H^{n-1}(L) \oplus H^{n-2}(L) \oplus \Pi H^{n-3}(L) \oplus H^{n-4}(L) \dots \quad (4.18)$$

From (4.17) it follows that considering map (4.13) on closed semidensities for arbitrary identifying symplectomorphisms and projecting value of this map on the superspace (4.18) we come to well-defined map

$$\hat{V}(L, \mathbf{s}) = w_n(L, \mathbf{s}) + [w_{n-1}](L, \mathbf{s}) + \ldots + [w_0](L, \mathbf{s}), \quad \text{if } \Delta^{\#}\mathbf{s} = 0 \quad (4.19)$$

($[w_k]$ is cohomology class of form w_k). The map $\hat{V}(L, \mathbf{s})$ is linear surjection map from the space $\mathcal{B}$ of closed semidensity on superspace (4.18).

By definition $\hat{V}(L, \mathbf{s})$ is $Can_{\rm adj}$-invariant map: it does not change under arbitrary canonical transformation adjusted to the surface L:

$$\text{if } \mathbf{s}_1 = F^*\mathbf{s}_2 \text{ where } F \in Can_{\rm adj}(L) \text{ then } \hat{V}(L, \mathbf{s}_1) = \hat{V}(L, \mathbf{s}_2)\,. \quad (4.20)$$

The opposite implication is obeyed for closed non-degenerate semidensities. Namely consider map (4.19) for the subset $\mathcal{B}_{\rm deg}$ of closed even non-degenerate semidensities (see 2.21). If $\hat{V}(L, \mathbf{s}_1) = \hat{V}(L, \mathbf{s}_2)$ for two arbitrary closed non-degenerate semidensities, then from statement c) of Proposition 4 it follows that there exists canonical transformation F adjusted to surface L such that $\mathbf{s}_1 = F^*\mathbf{s}_2$.

Now we analyze dependence of map (4.19) under a changing of even Lagrangian surface L. We study this point from more general point of view considering an action of group $Can(E)$ of all canonical transformations on maps (4.13) and (4.19).

Projecting superspace $\Omega^n(L)$ of top-degree forms on superspace $H^n(L)$ of corresponding cohomology classes we come from the map $\hat{V}(L, \mathbf{s})$ to the map

$$\hat{H}(L, \mathbf{s}) = [w](L, \mathbf{s}) = [w_n](L, \mathbf{s}) + \ldots + [w_0](L, \mathbf{s}) \quad (4.21)$$

that is defined on superspace $\mathcal{B}$ of closed semidensities and takes values in a superspace

$$H^*(L) = H^n(L) \oplus \Pi H^{n-1}(L) \oplus H^{n-2}(L) \oplus \Pi H^{n-3}(L) \oplus \ldots$$

From statement b) of Proposition 4 it follows that this map is $Can_H(E)$-invariant: it does not change under arbitrary canonical transformations generated by Hamiltonian:

$$\text{if } \mathbf{s}_1 = F^*\mathbf{s}_2 \text{ where } F \in Can_H(E) \text{ then } \hat{H}(L, F^*\mathbf{s}) = \hat{H}(L, \mathbf{s})\,. \tag{4.22}$$

The opposite implication is obeyed under the following restriction. Let $\mathbf{s}_1$ and $\mathbf{s}_2$ be two closed non-degenerated semidensities ($\mathbf{s}_1, \mathbf{s}_2 \in \mathcal{B}_{\rm deg}$) such that for the map (4.22) $\hat{H}(L, \mathbf{s}_1) = \hat{H}(L, \mathbf{s}_2)$. In this case $\mathbf{s}_2 = \mathbf{s}_1 + \Delta^\#\mathbf{r}$. If one-parametric family of closed semidensities $\mathbf{s}_t = \mathbf{s}_1 + t\Delta^\#\mathbf{r}$ $(0 \leq t \leq 1)$ belongs also to $\mathcal{B}_{\rm deg}$ then there exists canonical transformation F generated by Hamiltonian ($F \in Can_H(E)$) such that $F^*\mathbf{s}_1 = \mathbf{s}_2$. One comes to this transformation considering Hamiltonian (4.15). We note that in the special case when $Can_0(E) = Can_H(E)$, i.e. $H^1(M) = 0$ (see Proposition 3), these considerations lead to the statement of Theorem 5 in the paper [22].

Consider now the action of arbitrary canonical transformation on map (4.21).

From Proposition 3 and (4.22) it follows that under arbitrary canonical transformation the map $\hat{H}(L.\mathbf{s})$ have to be transformed under the action of the group $Can(E)/Can_H(E) = \Pi H^1(L) \ltimes \pi_0(Diff(L))$. Namely

$$\hat{H}(L, F^*\mathbf{s}) = [f]^* \left([\Psi] \lceil \hat{H}(L, \mathbf{s}) \right)\,, \tag{4.23}$$

where $[[\Psi], [f]]$ is an element of supergroup $\Pi H^1(L) \ltimes \pi_0(Diff(L))$ defined by the action of epimorphism (4.10a) on canonical transformation F and the operation $\lceil$ is defined for semidensities and corresponding differential forms by operations (3.10) and (3.11). The pull-back $[f]^*$ of equivalence class $[f]$ is well-defined, because pull-back f_0^* of diffeomorphism $f_0 \in Diff_0(L)$ acts identically on cohomological classes of differential forms.

On the other hand it follows from (4.14) that

$$\hat{H}(L, F^*\mathbf{s}) = (F|_L)^* \hat{H}(\widetilde{L}, \mathbf{s})\,, \tag{4.24}$$

where $\widetilde{L}$ is an image of Lagrangian surface L under canonical transformation F,

One can easy derive formulae (4.23) and (4.24) from (4.13) performing calculations in arbitrary Darboux coordinates adjusted to cotangent bundle structure of Lagrangian surface L (i.e. choosing arbitrary identifying symplectomorphism φ_L) and using decomposition formula (4.8).

It is useful to rewrite formulae (4.23) and (4.24) in components:

$$[w_k](L, F^*\mathbf{s}) = (F|_L)^*[w_k](\widetilde{L}, \mathbf{s}) = [f]^* \left(\sum_{p=0}^{k} \frac{1}{p!} \underbrace{[\Psi] \wedge \ldots \wedge [\Psi]}_{p \quad \text{times}} \wedge [w_{k-p}] \right)\,. \tag{4.25}$$

We note that if for a given pair $(L, \widetilde{L})$ of even Lagrangian surfaces canonical transformation F transforms L to $\widetilde{L}$ then cohomological class of odd valued one-form corresponding to the pair (F, L) by epimorphism (4.10a) is well-defined function of the pair $(L, \widetilde{L})$:

$$\Pi H^1(L) \ni [\Psi] = [\Psi](L, \widetilde{L}) . \tag{4.26}$$

The pull-back $(F|_L)^*$ of restriction $F|_L$ of canonical transformation F on Lagrangian surface L induces bijective map between differential forms and corresponding cohomological classes on surfaces L and $\widetilde{L}$. Using (4.25) we can compare cohomological classes of differential forms corresponding to a given closed semidensity for two different even Lagrangian surfaces. In particular, from (4.25) it follows that for arbitrary closed semidensity $\mathbf{s}$ and for arbitrary pair of closed Lagrangian surfaces $(L, \widetilde{L})$

$$[w_k](\widetilde{L}, \mathbf{s}) = 0 \quad \text{if} \quad [w_0](L, \mathbf{s}) = \ldots = [w_k](L, \mathbf{s}) = 0 ,$$

and in the case if cohomological class of one-form $[\Psi](L, \tilde{L})$ in (4.26) is equal to zero, then

$$[w_k](\widetilde{L}, \mathbf{s}) = 0 \quad \text{iff} \quad [w_k](L, \mathbf{s}) = 0 . \tag{4.27}$$

The simple but important consequence of these considerations is following: $[w_0]$-component of function $\hat{H}(L, \mathbf{s})$ (4.21) does not depend on canonical transformation and it is invariant constant on all Lagrangian surfaces.

Corollary 1 *To every closed semidensity* $\mathbf{s}$ *(*$\Delta^{\#}\mathbf{s} = 0$*) corresponds a positive constant* $c(\mathbf{s})$*. If in arbitrary Darboux coordinates*

$$\mathbf{s} = s(x, \theta)\sqrt{D(x, \theta)} = (\rho(x) + b^i(x)\theta_i + \ldots + c\theta_1\theta_2 \ldots \theta_n)\sqrt{D(x, \theta)}$$

then $c(\mathbf{s}) = |c|$*. This constant does not depend on the choice of Darboux coordinates and on the changing of density under arbitrary canonical transformation. This constant is equal (up to a sign) to cohomological class* $[w_0]$ *of zeroth order differential form corresponding to semidensity* $\mathbf{s}$ *on arbitrary even Lagrangian surface* L*. (A sign of* $c(\mathbf{s})$ *depends on orientation.)*

Note that $c(\mathbf{s})$ can be considered as integral of semidensity $\mathbf{s}$ over Lagrangian $(0.n)$-dimensional surface $x^1 = x_0^1, \ldots, x^n = x_0^n$: $c = \int_L \mathbf{s} = \int s(x_0, \theta) d^n\theta$.

4.3 Application to BV-geometry

Now using results obtained in this Section we analyze Statement 1 (see Introduction) of Batalin-Vilkovisky master equation.

Let $\mathbf{s}$ be an arbitrary closed semidensity in $\mathcal{B}_{\deg}$, i.e. non-degenerate semidensity that obeys BV-master equation (1.3b, 2.21).

For arbitrary Λ-point α in the odd symplectic supermanifold $E = E^{n.n}$ consider arbitrary closed even Lagrangian surface L such that this point belongs to this surface and choose arbitrary identifying symplectomorphism $\varphi_{_L}$, corresponding to this surface, i.e. atlas of Darboux coordinates adjusted to cotangent bundle structure of this Lagrangian surface. Consider on L differential form

$$w(L, \varphi_{_L}, \mathbf{s}) = w_n + w_{n-1} + \ldots + w_0 \tag{4.28}$$

defined by the map (4.13).

Locally all closed differential forms except zeroth-forms are exact and $[w_0] = \pm c(\mathbf{s})$ is invariant constant according to Corollary 1. Hence using statement c) of Proposition 4 one can find canonical transformation adjusted to this Lagrangian surface and correspondingly another identifying symplectomorphism $\varphi'_{_L}$ such that in (4.28) all differential forms w_k for $k = 1, \ldots, n-1$ vanish in a vicinity of the point α. Consider Darboux coordinates $z^A = \{x^1, \ldots, x^n,\ \theta_1, \ldots, \theta_n\}$ on E in a vicinity of this point from the atlas of Darboux coordinates corresponding to the identifying symplectomorphism $\varphi'_{_L}$ and by suitable "point" canonical transformation (2.14b) choose them in a way that differential form w_n is equal to $dx^1 \wedge \ldots \wedge dx^n$ in these coordinates. Thus we come to Darboux coordinates in a vicinity of the point α such that in these Darboux coordinates semidensity $\mathbf{s}$ has following appearance:

$$\mathbf{s} = s(x, \theta)\sqrt{D(x, \theta)} = (1 + c\theta_1\theta_2 \ldots \theta_n)\sqrt{D(x, \theta)}\,, \tag{4.29}$$

where c is equal up to sign to the invariant constant $c(\mathbf{s})$ corresponding to the semidensity $\mathbf{s}$ (see Corollary 1). The condition $c(\mathbf{s}) \neq 0$ is the obstacle to condition (1.3a).

Consider now the value of the map (4.19) on this Lagrangian surface:

$$\hat{V}(L, \mathbf{s}) = w_n + [w_{n-1}] + \ldots + [w_0]\,. \tag{4.30}$$

If $\hat{V}(L, \mathbf{s}) = w_n + c$, i.e. all cohomological classes $[w_k]$ for $k = 1, \ldots, n-1$ in (4.30) vanish on the surface L, then one can consider identifying symplectomorphism $\varphi_{_L}$ such that $\tau_L^{\#}(w_n + c) = \varphi_{_L}^{*}\mathbf{s}$ for the map (4.13). It means that there exists an atlas $\left[\{x^i_{(\alpha)}, \theta_{j(\alpha)}\}\right]$ of Darboux coordinates on $E^{n.n}$ adjusted to cotangent bundle structure of Lagrangian surface L such that in arbitrary coordinates from this atlas semidensity $\mathbf{s}$ is expressed by relation (4.29). Semidensity $\mathbf{s}$ has appearance $\sqrt{D(x, \theta)}$ in any Darboux coordinates from this atlas if invariant constant $c(\mathbf{s}) = 0$. In other words in this case supermanifold can be identified with ΠT^*L with volume form on ΠT^*L induced by volume form on L.

It follows from (4.23—4.27) that this statement holds for another even Lagrangian surface $\tilde{L}$ iff cohomological class $[\Psi]$ of one-form corresponding to a pair $(L, \tilde{L})$ of Lagrangian surfaces (see 4.26) is equal to zero. In particular this statement is irrelevant to a choice of Lagrangian surface if $H^1(M) = 0$. (M is underlying manifold for E.)

Now we analyze condition (1.3c) for even-nondegenerate semidensity $\mathbf{s} = \sqrt{d\mathbf{v}}$. From (2.19) it follows that condition (1.3c) means that function (2.17d) is equal to an odd constant ν, and $\Delta^{\#}\mathbf{s} = \nu\mathbf{s}$. One can see using correspondence between semidensities and differential forms that all solutions to this equation are following: $\mathbf{s} = \Delta^{\#}\mathbf{h} - \nu\mathbf{h}$, where $\mathbf{h}$ is an arbitrary semidensity. The odd constant $\nu \neq 0$ is the obstacle to condition (1.3b), if condition (1.3c) is obeyed.
We come to the

Corollary 2

Let $E = E^{n.n}$ be an odd symplectic supermanifold with connected orientable underlying manifold M and this supermanifold is provided with a volume form $d\mathbf{v}$, such that $\Delta^2_{d\mathbf{v}} = 0$. Then

1) *to the volume form $d\mathbf{v}$ corresponds the odd constant ν: $\Delta^{\#}\sqrt{d\mathbf{v}} = \nu\sqrt{d\mathbf{v}}$ and $\sqrt{d\mathbf{v}}$* $= \Delta^{\#}\mathbf{h} - \nu\mathbf{h}$ for some odd semidensity $\mathbf{h}$.

2) *If the odd constant ν is equal to zero, then the master-equation $\Delta^{\#}\sqrt{d\mathbf{v}} = 0$ holds for semidensity $\sqrt{d\mathbf{v}}$. In this case to the volume form $d\mathbf{v}$ corresponds non-negative constant $c = c(\sqrt{d\mathbf{v}})$ and there exists an atlas of Darboux coordinates on $E^{n.n}$ such that $d\mathbf{v} = (1 \pm 2c)D(x,\theta)$ in any coordinates from this atlas.*

3) *In the case if the constant $c(\mathbf{s}) = 0$ then, there exists an atlas of Darboux coordinates on E such that $d\mathbf{v} = D(x,\theta)$ in any coordinates from this atlas.*

4) *In the case if all cohomological classes of differential forms of degree less than n corresponding to the semidensity $\sqrt{d\mathbf{v}}$ on even Lagrangian surface L are equal to zero also, then there exists an atlas of Darboux coordinates on $E^{n.n}$ adjusted to cotangent bundle structure of L such that $d\mathbf{v} = D(x,\theta)$ in any coordinates from this atlas, i.e. E can be identified with ΠT^*L with volume form on ΠT^*L induced by volume form on L.*

This statement holds for another $(n.0)$-dimensional Lagrangian surface $\widetilde{L}$ if cohomological class of odd valued one-form $[\Psi](L,\widetilde{L})$ is equal to zero.

This Corollary removes uncorrectness of the considerations about equivalence of conditions (1.3a), (1.3b) and (1.3c) in the Statement 1 of Introduction, which was done in [15] and [22]. On the other hand some statements of this Proposition in non explicit way were contained in the statements of Lemma 4 and Theorem 5 of the paper [22].

5. Invariant densities on surfaces

First we recall shortly the problem of construction of invariant densities in sympelctic (super)manifolds. Then we consider explicit formulae for the odd invariant semidensity on non-degenerate surfaces of codimension (1.1) embedded in an odd symplectic supermanifold E provided with a volume form $d\mathbf{v}$ ([12,13]). We consider this semidensity as a kind of pull-back of semidensity $\mathbf{s}$ from the ambient odd symplectic supermanifold on embedded (1.1)-codimensional surfaces in the case if $\mathbf{s} = \sqrt{d\mathbf{v}}$. Using this construction for the semidensity $\Delta^{\#}\sqrt{d\mathbf{v}}$ we will construct the new densities on embedded non-degenerated surfaces.

In the case if we consider a volume form not only on the space (superspace) but on arbitrary embedded surfaces we come to the concept of densities on embedded surfaces. The density of weight σ and rank k on embedded surfaces is a function $A(z, \frac{\partial z}{\partial\zeta}, \ldots, \frac{\partial^k z}{\partial\zeta\ldots\partial\zeta})$ that is defined on parameterized surfaces $z(\zeta)$, depends on first k derivatives of $z(\zeta)$ and is multiplied on the σ-th power of determinant (Berezinian) of surface reparametrization.

A density of a weight σ defines on every given surface σ-th power of volume form. Such a concept of density is very useful in supermathematics where the notion of differential forms as integration objects is ill-defined. It was elaborated by A.S.Schwarz, particularly for analyzing supergravity Lagrangians [20, 9, 21].

In usual mathematics, for every $2k$-dimensional surface C^{2k} embedded in a symplectic space, so called Poincaré-Cartan integral invariants (invariant volume forms on embedded surfaces) are given by the formula

$$\int_{C^{2k}} \underbrace{w \wedge \dot{s} \wedge w}_{k-\text{times}} = \int \sqrt{\det\left(\frac{\partial x^\mu(\xi)}{\partial \xi^i} w_{\mu\nu} \frac{\partial x^\nu(\xi)}{\partial \xi^j}\right)} d^{2k}\xi \,, \tag{5.1}$$

where a non-degenerate closed two-form $w = w_{\mu\nu}dx^\mu \wedge dx^\nu$ defines symplectic structure, and functions $x^\mu = x^\mu(\xi^i)$ define some parameterizations of the surface C^{2k}.

In supermathematics one can consider even and odd symplectic structures on supermanifold generated by even and odd non-degenerate closed two-forms respectively [6,18,19].

In the case of an even symplectic supermanifold, the l.h.s. of (5.1) is ill-defined but the r.h.s. of this formula can be straightforwardly generalized, by changing determinant on the Berezinian (superdeterminant). The properties of the integral invariant do not change drastically. In particular one can prove that the integrand in (5.1) (the density of the weight $\sigma = 1$ and of the rank $k = 1$) is locally total derivative and all invariant densities on surfaces are exhausted by (5.1) as well as in the case of usual symplectic structure [16, 1].

The situation is less trivial in the case of an odd symplectic supermanifold. Formula (5.1) cannot be generalized in this case because transformations preserving odd symplectic structure do not preserve any volume form . One can consider invariant densities only in an odd symplectic supermanifold provided with a volume form.

The problem of the existence of invariant densities on non-degenerate surfaces embedded in an odd symplectic supermanifold provided with a volume form was studied in [12, 13]. In particularly it was proved that there are no invariant densities of the rank $k = 1$ (except of the volume form itself), and invariant semidensity of the rank $k = 2$ that is defined on non-degenerate surfaces of the codimension (1.1) was obtained. We briefly expose here its construction.

The surface embedded in symplectic supermanifold is called non-degenerate if the sympelctic structure of the supermanifold generates the symplectic structure on the embedded surface also, i.e. if the pull-back of the symplectic 2-form on the surface is non-degenerate 2-form. This symplectic structure on an embedded surface is called induced symplectic structure.

Let $\{z^A\}$ be Darboux coordinates on an odd symplectic supermanifold $E = E^{n.n}$ provided with volume form $d\mathbf{v} = \rho(z)Dz$. It is convenient in this section to use for Darboux coordinates notations $z^A = (x^\mu, \theta_\mu)$, $(\mu = (0,i) = (0,1,\dots,n-1)$, $i = (1,\dots,n-1))$. Let $z(\zeta)$ be an arbitrary parameterization of an arbitrary non-degenerate surface of codimension (1.1), embedded in E. $(\zeta = (\xi^i, \eta_j)$, ξ^i and η_j are even and odd parameters respectively, $(i,j = 1,\dots,n-1))$. The invariant semidensity of the rank $k=2$ (depending on first and second derivatives of $z(\zeta)$) is given by the following formula [12]:

$$A\left(z(\zeta), \frac{\partial z}{\partial \zeta}, \frac{\partial^2 z}{\partial \zeta \partial \zeta}\right)\sqrt{D\zeta} =$$

$$\left(\Psi^A \frac{\partial \log \rho(z)}{\partial z^A} - \Psi^A \Omega_{AB} \frac{\partial^2 z^B}{\partial \zeta^\alpha \partial \zeta^\beta} \Omega^{\alpha\beta}(z(\zeta))(-1)^{p(z^B(\zeta^\alpha+\zeta^\beta)+\zeta^\alpha}\right)\sqrt{D\zeta}\,, \tag{5.2}$$

where $\Omega_{AB} dz^A dz^B$ is the two-form defining the odd sympelctic structure on $E^{n.n}$ and $\Omega^{\alpha\beta}$ is the tensor inversed to the two-form that defines induced symplectic structure on the surface. The vector field $\mathbf{\Psi} = \Psi^A \frac{\partial}{\partial z^A}$ is defined as follows: one have to consider the pair of vectors $(\mathbf{H}, \mathbf{\Psi})$, $\mathbf{H}$ even and $\mathbf{\Psi}$ odd that are symplectoorthogonal to the surface and obey the following conditions:

$$\Omega(\mathbf{H}, \mathbf{\Psi}) = 1,\ \Omega(\mathbf{\Psi}, \mathbf{\Psi}) = 0 \quad (\text{symplectoorthonormality conditions})\,, \tag{5.3}$$

$$d\mathbf{v}\left(\left\{\frac{\partial z}{\partial \zeta}\right\}, \mathbf{H}, \mathbf{\Psi}\right) = 1 \quad (\text{volume form normalization conditions})\,. \tag{5.4}$$

These conditions fix uniquely the vector field $\mathbf{\Psi}$. (See for details [12]).

The explicit expression for this semidensity was calculated in [13] in terms of dual densities: If (1.1)-codimensional surface C is given not by parameterization, but by the equations $f = 0, \varphi = 0$, where f is an even function and φ is an odd function then to the semidensity (5.2) there corresponds the dual semidensity:

$$\tilde{A}\Big|_{f=\varphi=0} = \frac{1}{\sqrt{\{f,\varphi\}}}\left(\Delta_{d\mathbf{v}} f - \frac{\{f,f\}}{2\{f,\varphi\}}\Delta_{d\mathbf{v}}\varphi - \frac{\{f,\{f,\varphi\}\}}{\{f,\varphi\}} - \frac{\{f,f\}}{2\{f,\varphi\}^2}\{\varphi,\{f,\varphi\}\}\right)\,. \tag{5.5}$$

One can check that r.h.s. of (5.5) restricted by conditions $f = \varphi = 0$ is multiplied by the square root of the corresponding Berezinian (superdeterminant) under the transformation $f \to af + \alpha\varphi$, $\varphi \to \beta f + b\varphi$, which does not change the surface C [13].

This invariant semidensity takes odd values. It is an exotic analogue of Poincaré–Cartan invariant: the corresponding density (the square of this odd semidensity) is equal to zero, so it cannot be integrated nontrivially over surfaces. On the other hand this semidensity can be considered as an analog of the mean curvature of hypersurfaces in the Riemanian space [12].

This odd semidensity in an odd symplectic supermanifold is unique (up to multiplication by a constant) in the class of densities of the rank $k = 2$ that are defined on non-degenerated surfaces of arbitrary dimension [13]. This means that one have to search non-trivial integral invariants (invariant densities of weight $\sigma = 1$) in higher order derivatives (rank $k \geq 3$). Tedious calculations, which lead to the construction of the odd invariant semidensity in the papers [12,13] did not give hope to go further for finding them, using the technique used in these papers.

Now we develop another approach rewriting the semidensity (5.2) straightforwardly via the semidensity $\sqrt{d\mathbf{v}}$ on the ambient odd symplectic supermanifold $E = E^{n.n}$.

Consider for every given non-degenerate surface C of codimension (1.1) embedded in odd symplectic supermanifold E Darboux coordinates such that in these Darboux coordinates the surface C locally is given by equations

$$x^0 = \theta_0 = 0\,. \tag{5.6}$$

We call these Darboux coordinates adjusted to the surface C. (The existence of Darboux coordinates obeying these conditions can be proved using technique considered in Appendices 2 and 3).

If $\{x^\mu, \theta_\nu\}$ are Darboux coordinates in E adjusted to the surface C, then $\{x^i, \theta_j\}$ are Darboux coordinates on the surface $C^{n-1.n-1}$ w.r.t. the induced symplectic structure ($\mu, \nu = 0, \ldots, n-1$, $i, j = 1, \ldots, n-1$).

Consider a semidensity (5.2) on arbitrary non-degenerated surface $C = C^{n-1.n-1}$ of codimension (1.1) in Darboux coordinates (5.6) adjusted to this surface. Conditions of symplectoorthonormality in (5.3) give that $\mathbf{H} = \frac{1}{a}\frac{\partial}{\partial x^0} + \beta\frac{\partial}{\partial\theta_0}$ and $\mathbf{\Psi} = a\frac{\partial}{\partial\theta_0}$, where a is even and β is odd. The condition (5.4) of the volume form normalization gives that

$$a = \sqrt{\rho}\,\mathrm{Ber}^{1/2}\left(\frac{\partial(x^i,\theta_j)}{\partial(\xi^i,\eta_j)}\right)\,,$$

where a volume form $d\mathbf{v}$ is equal to $\rho(x,\theta)D(x,\theta)$ and $\zeta = (\xi^i, \eta_j)$ are parameters ($x^0 = \theta_0 = 0$, $x^i = x^i(\xi,\eta)$, $\theta_j = \theta_j(\xi,\eta)$).

Hence the semidensity (5.2) on a surface (5.6) is reduced to

$$A\left(z(\zeta), \frac{\partial z}{\partial\zeta}, \frac{\partial^2 z}{\partial\zeta\partial\zeta}\right)\sqrt{D\zeta} = a\frac{\partial\log\rho}{\partial\theta_0}\sqrt{D\zeta} = 2\frac{\partial\sqrt{\rho}}{\partial\theta_0}\sqrt{D(x^i,\theta_j)}\,.$$

We come to the following statement

Theorem *To every semidensity* $\mathbf{s}$ *in the odd symplectic supermanifold* E *corresponds semidensity* $\mathcal{K}(\mathbf{s})$ *of an opposite parity defined on non-degenerated* (1.1)*-codimensional surfaces embedded in this supermanifold.*

If semidensity $\mathbf{s}$ *is given by expression* $\mathbf{s} = s(x,\theta)\sqrt{D(x,\theta)}$ *in Darboux coordinates* $\{x^\mu, \theta_\nu\} = \{x^0, x^i, \theta_0, \theta_j\}$ *adjusted to given non-degenerate surface* C *of codimension* (1.1) *(*$x^0|_C = \theta_0|_C = 0$*), then semidensity* $\mathcal{K}(\mathbf{s})$ *on this surface in these Darboux coordinates is given by the following expression*:

$$\mathcal{K}(\mathbf{s})\big|_C = \frac{\partial s(x^\mu, \theta_\nu)}{\partial \theta_0}\Big|_{x^0=\theta_0=0} \sqrt{D(x^i, \theta_j)}\,, \tag{5.7}$$

where $D(x^\mu, \theta_\nu)$ is coordinate volume form on the supermanifold E and $D(x^i, \theta_j)$ is coordinate volume form on the surface C.

The considerations above lead to the statement of this Theorem for semidensities related with a volume form on an odd symplectic supermanifold ($\mathbf{s} = \sqrt{d\mathbf{v}}$), i.e. for even non-degenerate even semidensities. Continuity considerations lead to the fact that the formula (5.7) is well-defined for an arbitrary semidensity e.g. for an odd semidensity, when the corresponding volume form is equal to zero.

Alternatively one can prove this Theorem checking in a same way as for (2.12) that the semidensity in r.h.s. of (5.7) is well defined. For example consider canonical transformation that has the following appearance in Darboux coordinates adjusted to surface C:

$$\begin{cases} \tilde{x}^0 = \tilde{x}^0(x^0, \theta_0)\,,\ \tilde{\theta}_0 = \theta_0(x^0, \theta_0) \\ \tilde{x}^i = x^i\,, \tilde{\theta}_i = \theta_i \end{cases}$$

One can see that these canonical transformations are exhausted by transformations $\tilde{x}^0 = f(x^0)$, $\tilde{\theta}_0 = \beta(x^0) + \theta_0/f_x$, $\tilde{x}^i = x^i, \theta_i = \theta_i$, where $f(x)$ and $\beta(x)$ are even-valued and odd valued functions on x respectively. Hence for transformation of adjusted coordinates $\tilde{x}^0 = f(x^0)$, $\tilde{\theta}_0 = \theta/f_x$. Obviously r.h.s. of formula (5.7) transforms as semidensity under this transformation. This is the central point of the construction (5.7) and also of (5.2) (see for details [12]).)

We can consider a semidensity $\mathcal{K}(\mathbf{s})$ in (5.7) as a kind of pull-back of semidensity $\mathbf{s}$ on C, but this construction does not obey a condition of transitivity for pull-back: consider arbitrary $(k.k)$-dimensional non-degenerate surface embedded in $E^{n.n}$ and include it in a flag of non-degenerated surfaces:

$$Y^{k.k} \hookrightarrow Y^{k+1.k+1} \ldots \hookrightarrow Y^{n-1.n-1} \hookrightarrow E^{n.n}\,. \tag{5.8}$$

then one can consider semidensity $\mathcal{K}(\ldots\mathcal{K}(\mathbf{s})\ldots)$ on $Y^{k.k}$ corresponding to the semidensity $\mathbf{s}$ depending on flag (5.8).

The statement of Theorem allows us to construct semidensity on embedded surfaces via odd semidensities on the ambient supermanifold, which cannot be yielded from volume forms.

In an odd sympelctic supermanifold provided with a volume form $d\mathbf{v}$ on (1.1)-codimensional non-degenerate surfaces except an odd semidensity $\mathcal{K}(\sqrt{d\mathbf{v}})$, that is nothing but semidensity (5.2), one can consider also an even semidensity $\mathcal{K}(\Delta^{\#}\sqrt{d\mathbf{v}})$ corresponding to an odd semidensity $\Delta^{\#}\sqrt{d\mathbf{v}}$. The semidensity $\mathcal{K}(\Delta^{\#}\sqrt{d\mathbf{v}})$ cannot be represented (5.2)-like because the square of the odd semidensity $\Delta^{\#}\sqrt{d\mathbf{v}}$ is equal to zero.

We note that for semidensities $\mathcal{K}(\Delta^{\#}\mathbf{s})$ and $\mathcal{K}(\mathbf{s})$ for arbitrary (1.1)-codimensional surface C the following condition is obeyed:

$$\mathcal{K}(\Delta^{\#}\mathbf{s})\Big|_C = -\widetilde{\Delta^{\#}}\mathcal{K}(\mathbf{s})\Big|_C\,, \tag{5.9}$$

where $\widetilde{\Delta^{\#}}$ is $\Delta^{\#}$-operator on surface C w.r.t. induced symplectic structure. This relation can be immediately checked in Darboux coordinates (5.6) adjusted to the surface C.

The semidensities $\mathcal{K}(\sqrt{d\mathbf{v}})$ and $\mathcal{K}(\Delta^{\#}\sqrt{d\mathbf{v}})$ can be integrated over Lagrangian subsurfaces in C, according (4.16).

On the other hand one can consider the following non-trivial densities of weight $\sigma = 1$ constructed via the semidensities $\mathcal{K}(\sqrt{d\mathbf{v}})$ and $\mathcal{K}(\Delta^{\#}\sqrt{d\mathbf{v}})$:

$$P_0 = \mathcal{K}^2(\Delta^{\#}\sqrt{d\mathbf{v}}) \quad \text{and} \quad P_1 = \mathcal{K}(\sqrt{d\mathbf{v}})\mathcal{K}(\Delta^{\#}\sqrt{d\mathbf{v}})\,. \tag{5.10}$$

The density P_0 takes even values, the density P_1 takes odd values. In general case these densities give non-trivial integration objects (volume forms) over non-degenerated (1.1)-codimensional surfaces embedded in an odd symplectic supermanifold with volume form $d\mathbf{v}$.

The densities P_0 and P_1 have rank $k = 4$ (i.e. depend on derivatives of the parameterization $z(\zeta)$ up fourth order). It follows from the fact that according to (5.9) the semidensity $\mathcal{K}(\Delta^{\#}\sqrt{d\mathbf{v}})$ has the rank $k = 4$, because the semidensity $\mathcal{K}(\sqrt{d\mathbf{v}})$ has the rank $k = 2$. This is hidden in representation (5.7), where the function $\rho(z)$ corresponding to the volume form in adjusted coordinates depends non-explicitly on derivatives of surface parameterization $z(\zeta)$.

Finally we consider a simple example of these constructions and their relations with differential forms.

Let $E^{3.3}$ be a superspace associated to cotangent bundle of 3-dimensional space E^3, $E^{3.3} = \Pi T^*E^3$. We assume that coordinates x^0, x^1, x^2 are globally defined on E^3. We consider on E^3 the differential form

$$w = -dx^0 \wedge dx^1 \wedge dx^2 + b_0 dx^0 + b_1 dx^1 + b_2 dx^2\,.$$

According to (3.4) a semidensity

$$\mathbf{s} = \tau^{\#}(w) = (1 + b_0\theta_1\theta_2 + b_1\theta_2\theta_0 + b_2\theta_0\theta_1)\sqrt{D(x^0, x^1, x^2, \theta_0, \theta_1, \theta_2)}$$

in ΠT^*E corresponds to this differential form. Let C be a surface in $E^{3.3}$ that is defined by equations $x^0 = \theta_0 = 0$ and E^2 is subspace in E^3 defined by equation $x^0 = 0$. (2.2)-dimensional superspace C provided with coordinates $x^1, x^2, \theta_1, \theta_2$ can be identified with superspace ΠT^*E^2 associated with cotangent bundle T^*E^2 of subspace E^2.

Then the value of the odd semidensity $\mathcal{K}(\mathbf{s})$ on $C = \Pi T^*E^2$ is equal to $(b_2\theta_1 - b_1\theta_2)\sqrt{D(x^1, x^2, \theta_1, \theta_2)}$. This semidensity corresponds to differential form $b_1dx^1 + b_2dx^2$, the pull-back of w on E^2. The value of even semidensity $\mathcal{K}(\Delta^{\#}\mathbf{s})$ on C is equal to $(\partial_2 b_1 - \partial_1 b_2)\sqrt{D(x^1, x^2, \theta_1, \theta_2)}$. This corresponds to differential form $d(b_1dx^1 + b_2dx^2) = (\partial_2 b_1 - \partial_1 b_2)dx^1 \wedge dx^2$, the pull-back of dw on E^2.

The even density (volume form) on M is equal to $P_0 = (\partial_2 b_1 - \partial_1 b_2)^2 D(x^1, x^2, \theta_1, \theta_2)$ and the odd density $P_1 = (\partial_2 b_1 - \partial_1 b_2)(b_1\theta_2 - b_2\theta_1)D(x^1, x^2, \theta_1, \theta_2)$.

6. Discussion

The definition (2.7) of the $\Delta_{d\mathbf{v}}$-operator is applicable not only for a symplectic supermanifold but also for a Poisson supermanifold (provided with a volume form) even if corresponding Poisson bracket is degenerate. Is it possible to define $\Delta^{\#}$-operator on semidensities in odd Poisson supermanifold? This question is studied in [17]. It turns out that on general odd Poisson supermanifold there is a canonical Δ-operator on semidensities depending only on the class of a volume form modulo the action of a natural "master" groupoid (see [17]).

We note also that from relations (2.10) it follows that one can express odd Poisson bracket via the operator $\Delta_{d\mathbf{v}}$. Moreover every second order odd differential operator $\hat{A}$ on functions on a supermanifold obeying the condition $\hat{A}^2 = 0$ defines Poisson structure via relations (2.10). This approach was elaborated by I.A. Batalin and I.V. Tyutin [2]. (The exhaustive study of these questions giving in particular a complete description of the BV operators can be found in [26].)

On an odd symplectic supermanifolds the integration theory on surfaces interplays with symplectic geometry. Using our approach one can consider integrands (differential forms) of in terms of semidensities corresponding to these differential forms. In this case symmetry transformations of the corresponding functionals are not exhausted by transformations induced by diffeomorphisms of underlying space. General canonical transformations of supermanifold induce mixing of corresponding differential forms with different degrees.

In Sections 3 and 4 we investigated relations between semidensities on an odd symplectic supermanifold and differential forms on even Lagrangian surfaces. It is interesting to generalize these results to the case when Lagrangian surface is $(n-k.k)$-dimensional for $k \neq 0$ using the analysis of arbitrary Lagrangian surfaces performed in the paper [22]. In this case there exist analogies of differential forms considered as integration objects. (See [7, 1] and for a more detailed analysis [27, 25].) For example if ΠT^*M is $(r+s.r+s)$-dimensional supermanifold associated with the cotangent bundle of an $(r.s)$-dimensional supermanifold M then considering analogue of formula (3.4a) we obtain relations between semidensities in ΠT^*M and the so called pseudodifferential forms: functions on the supermanifold ΠTM. Pseudodifferential forms are well-defined integration objects over supermanifold and embedded surfaces [7, 1, 27, 25]. In this way it is possible to come to an analogue of the map (4.16) (see [22], [15]). It is interesting to construct analogues of the maps (4.19) and (4.21) for $(n-k.k)$-dimensional Lagrangian surfaces.

We note that our considerations in subsections 4.2 and 4.3 overlap partially with some results of the paper [22]. A distinctive feature of our approach is the use of semidensities where a calculus analogous to the calculus of differential forms arises. In particularly this leads to the statements in Corollary 2. Also, by using Λ-points we come to the difference between supergroups $Can_0(E)$ and $Can_H(E)$.

We hope that considerations presented in Section 5 of this paper can be generalized for constructing densities depending on higher order derivatives for surfaces of arbitrary dimension embedded in an odd symplectic supermanifold provided with a volume form and for finding a complete set of local invariants of this geometry. In particular, from considerations which lead to Theorem follows that if $k(p)$ is the rank of a non-trivial invariant densities on non-degenerated surfaces of codimension $(p.p)$, then $k(2) \geq 5$ and $k(p+1) > k(p)$.

In [12] some relations of the semidensity (5.2, 5.7) with mean curvature in Riemanian geometry were indicated. It is interesting to analyze these relations in terms of geometry of semidensities presented in this paper.

Densities presented in formula (5.10) are needed to be investigated in greater detail. Particularly one have to present explicit formulae for them and consider the corresponding functionals over surfaces. These functionals are equal to zero in the special case if the volume form in the ambient odd symplectic supermanifold obeys the BV-master equation. Are Euler-Lagrange equations for these functionals satisfied identically in a general case, as for the usual Poincare-Cartan integral invariants (5.1)?

Results presented in Section 5 strongly indicate that there exists non-trivial geometry in an odd symplectic supermanifold provided with a volume form.

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Appendix 1. Λ-points on Supermanifolds

Let $\left[\{x^i_{(\alpha)}\}\right]$ be a smooth atlas of coordinates on m-dimensional manifold M^m, where coordinates $\{x^i_{(\alpha)}\}$ are defined on domain U_α and $x^i_{(\alpha)} = \Psi^i_{\alpha\beta}(x_{(\beta)})$ are transition functions. Consider an atlas $\left[\{x^i_{(\alpha)}, \theta^j_{(\alpha)}\}\right]$, where odd variables $\{\theta^j_{(\alpha)}\}$ $(j = 1, \dots, n)$ are generators of Grassmann algebra and transition functions

$$\begin{cases} x^i_{(\alpha)} = \widetilde{\Psi}^i_{\alpha\beta}(x_{(\beta)}, \theta_{(\beta)}) \\ \theta^j_{(\alpha)} = \Phi^j_{\alpha\beta}(x_{(\beta)}, \theta_{(\beta)}) \end{cases} \tag{Ap1.1}$$

obey the following properties:

1) they are parity preserving, i.e. $p(\widetilde{\Psi}_{\alpha\beta}) = 0, p(\Phi_{\alpha\beta}) = 1$, where $p(x^i) = 0, p(\theta^j) = 1$,

2) $\widetilde{\Psi}_{\alpha\beta}(x_{(\beta)}, \theta_{(\beta)})\big|_{\theta^j=0} = \Psi_{\alpha\beta}(x_{(\beta)})$ and $\partial\Phi^j/\partial\theta^i_{(\beta)}$ are inverting matrices.

Coordinates $\{x^i_{(\alpha)}, \theta^j_{(\alpha)}\}$ define $(m.n)$-dimensional superdomain $\hat{U}^{m.n}_{(\alpha)}$ with underlying domain $U^m_{(\alpha)}$. Pasting formulae (Ap1.1) define $(m.n)$-dimensional supermanifold with underlying manifold M^m. In this definition of supermanifold which belongs to F.Berezin and D.Leites (see [6] and [19]) a supermanifold "has no points".

If E is supermanifold and Λ is an arbitrary Grassmann algebra one can construct a set E_Λ of Λ-points of supermanifold E. For example if $E^{m.n}$ is superdomain with underlying domain M^m, we define E_Λ as a set of rows $(a^1, \dots, a^m, \alpha^1, \dots, \alpha^n)$, where $a^1, \dots, a^m$ are arbitrary even elements and $\alpha^1, \dots, \alpha^n$ are arbitrary odd elements of Grassmann algebra Λ and $(m(a^1), \dots, m(a^m)) \in M^m$, where m is a standard homomorphism of Λ on IR. A map of superdomains generates a map of corresponding sets of Λ-points. Thus one comes to definition of a set E_Λ for arbitrary supermanifold E. To every parity preserving homomorphism ρ: $\Lambda \to \Lambda'$ of Grassmann algebras one can naturally assign a map $\tilde{\rho}_{_E}$: $E_\Lambda \to E_{\Lambda'}$. If ρ: $\Lambda \to \Lambda'$ and ρ': $\Lambda' \to \Lambda''$ are two parity preserving homomorphisms, then $\widetilde{(\rho \circ \rho')}_{_E} = \tilde{\rho_{_E}} \circ \tilde{\rho'_{_E}}$. Supermanifold can be considered as functor on the category of Grassmann algebras taking values in category of sets.

This definition of supermanifolds is used in the paper. It was suggested and widely used by A.S. Schwarz [21]. It makes possible to use a language of "points" and is more convenient for supergeometry and in applications in theoretical physics.

In terms of Λ-points one can easy to generalize the standard geometrical definitions on supercase [21]. For example

1. A map F from supermanifold E in supermanifold N can be considered as a functor from category $\{\Lambda\}$ of Grassmann algebras to category $\{F_\Lambda\}$, where for every Grassmann algebra Λ F_Λ is a map from the set E_Λ to the set N_Λ such that $F_{\Lambda'} \circ \tilde{\rho_E} = \tilde{\rho_N} \circ F_\Lambda$ for every parity preserving homomorphism ρ: $\Lambda \to \Lambda'$.

2. The action of supergroup G on supermanifold E can be considered as a functor that assigns to every Grassmann algebra the pair $[G_\Lambda, E_\Lambda]$ where G_Λ is a group of Λ-points of supergroup G, that acts on the set E_Λ of Λ-points of supermanifold E.

Appendix 2. A simple proof of Darboux Theorem for odd symplectic structure

Using nilpotency of odd variables one can directly prove Darboux theorem for an odd symplectic supermanifold presenting finite recurrent procedure for constructing Darboux coordinates starting from arbitrary coordinates.

Let $\{\quad,\quad\}$ be odd non-degenerated Poisson bracket (2.1) corresponding to the symplectic structure. According to (2.1) for arbitrary two functions f and g

$$\begin{array}{c}\{f,g\} = \frac{\partial f}{\partial x^i}\{x^i,x^j\}\frac{\partial g}{\partial x^j} + \frac{\partial f}{\partial x^i}\{x^i,\theta_j\}\frac{\partial g}{\partial \theta_j} + (-1)^{p(f)+1}\frac{\partial f}{\partial \theta_i}\{\theta_i,x^j\}\frac{\partial g}{\partial x^j} \\ +(-1)^{p(f)+1}\frac{\partial f}{\partial \theta_i}\{\theta_i,\theta_j\}\frac{\partial g}{\partial \theta_j}\end{array} \qquad \text{(Ap2.1)}$$

and Jacoby identities (2.3) are obeyed.

For given arbitrary coordinates $\{x^1,\dots,x^n,\theta_1,\dots,\theta_n\}$ denote by

$$E^{ij}(x,\theta) = \{x^i,x^j\}\,, F_{ij}(x,\theta) = \{\theta_i,\theta_j\}\,, A^i_j(x,\theta) = \delta^i_j + P^i_j(x,\theta) = \{x^i,\theta_j\}\,. \qquad \text{(Ap2.2)}$$

From definition of symplectic structure it follows that $E^{ij} = E^{ji}$, $F_{ij} = -F_{ji}$ are odd-valued matrices taking values in Grassmann algebra Λ and $A^i_j(x,\theta)$ is even non-degenerate matrix taking values in Grassmann algebra Λ. In Darboux coordinates matrices E^{ij}, F_{ij} and P^i_j have to be equal to zero.

First of all we note that in the case if for coordinates $\{x^i,\theta_j\}$ the conditions

$$E^{ik}(x,\theta) = 0\,,\quad P^i_k(x,\theta) = 0 \qquad \text{(Ap2.3)}$$

are obeyed then Jacoby identities $\{x^m,\{\theta_i,\theta_j\}\} + \text{cycl. permut.} = 0$ imply that F_{ij} do not depend on θ and Jacoby identities $\{\theta_i,\{\theta_j,\theta_m\}\} + \text{cycl. permut.} = 0$ imply the condition $\partial_i F_{jm}(x) + \partial_j F_{mi}(x) + \partial_m F_{ij}(x) = 0$. (In other words two-form $F_{ij}(x)dx^i\wedge dx^j$ is closed).

Locally it means that there exist functions $A_i(x)$ such that $F_{ij}(x) = \partial_i A_j(x) - \partial_j A_i(x)$. Under transformation $\theta_i \to \theta_i + A_i(x)$, $F_{ij}(x)$ transform to zero also and we come to Darboux coordinates.

Thus we have to find transformation from arbitrary coordinates to new coordinates such that in new coordinates conditions (Ap2.3) will be obeyed.

Consider a set $\mathcal{M}$ of all coordinates $\{x^i, \theta_j\}$. Denote by $\mathcal{M}_{(p.q)}$ a subset of $\mathcal{M}$ such that for coordinates $\{x^i, \theta_j\}$ belonging to the subset $\mathcal{M}_{(p.q)}$ the following conditions are obeyed for matrices $E^{ij}(x,\theta)$ and $P^i_j(x,\theta)$ in (Ap2.2):

$$E^{ij}(x,\theta) = O(\theta^p)\,, \quad P^i_j(x,\theta) = O(\theta^q)\,. \tag{Ap2.4}$$

$\mathcal{M}_{0.0} = \mathcal{M}$ and condition $\{x^i, \theta_j\} \in \mathcal{M}_{n+1.n+1}$ means that relations (Ap2.3) are obeyed for these coordinates, because $\theta_{i_1} \dots \theta_{i_k} = 0$ if $k \geq n+1$.

Consider four maps $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4$ defined on the set $\mathcal{M}$ of coordinates, such that these maps obey the following conditions:

$$\begin{gathered} \mathcal{F}_1 \text{ maps } \mathcal{M}_{r.0} \text{ in } \mathcal{M}_{r.1} \text{ for } r = 0, 1, \dots \quad , \\ \mathcal{F}_2 \text{ maps } \mathcal{M}_{0.1} \text{ in } \mathcal{M}_{1.0} \ , \\ \mathcal{F}_3 \text{ maps } \mathcal{M}_{r.1} \text{ in } \mathcal{M}_{r+1.1} \text{ for } r \geq 1\,, \\ \mathcal{F}_4 \text{ maps } \mathcal{M}_{n+1.r} \text{ in } \mathcal{M}_{n+1.r+1} \text{ for } r \geq 1\,. \end{gathered} \tag{Ap2.5}$$

Provided conditions (Ap2.5) are obeyed the map $\mathcal{F}_4^n \circ \mathcal{F}_3^n \circ \mathcal{F}_1 \circ \mathcal{F}_2 \circ \mathcal{F}_1$ transforms arbitrary coordinates to coordinates that belong to subset $\mathcal{M}_{n+1.n+1}$, i.e. conditions (Ap2.3) are obeyed for transformed coordinates.

Now we present maps $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4$ obeying conditions (Ap2.5).

1. Definition of the map $\mathcal{F}_1$:

$$\mathcal{F}_1(\{x^i, \theta_j\}) = \{\tilde{x}^i, \tilde{\theta}_j\}, \quad \text{where} \quad \tilde{x}^i = x^i, \tilde{\theta}_j = \theta_m (A^{-1})^m_j\,, \tag{Ap2.6}$$

where matrix A^{-1} is inverse to the matrix A defined by relations (Ap2.2) for coordinates $\{x^i, \theta_j\}$. It is easy to see from (Ap2.1) that map (Ap2.6) obeys condition (Ap2.5).

2. Definition of the map $\mathcal{F}_2$:

$$\mathcal{F}_2(\{x^i, \theta_j\}) = \{\tilde{x}^i, \tilde{\theta}_j\}, \quad \text{where} \quad \tilde{x}^i = x^i - \theta_m R^{mi}, \quad \tilde{\theta}_j = \theta_j\,, \tag{Ap2.7}$$

where symmetrical odd-valued matrix R is solution to matrix equation

$$2R + RFR = E\,, \quad (R^{ij} = R^{ji}) \tag{Ap2.8}$$

and matrices E and F for coordinates $\{x^i, \theta_j\}$ are defined by (Ap2.2).

The solution to this equation is well-defined because elements of symmetric matrix E and antisymmetric matrix F take odd values in Grassmann algebra Λ. R is given by finite power series $R = \frac{E}{2} - \frac{EFE}{8} + \ldots$ containing less than $[\frac{n^2}{2}]$ terms. One can present explicit solution to equation (Ap2.8):

$$R = \sum_{k=0}^{\frac{n(n-1)}{2}} c_k (EF)^k E, \quad \text{where} \quad \sum_{k=0}^{\infty} c_k t^k = \frac{\sqrt{1+t}-1}{t}\,. \tag{Ap2.9}$$

Now it follows from (Ap2.1) and (Ap2.8) that under transformation (Ap2.7) matrix $E^{ij} = \{x^i, x^j\}$ transforms to the matrix $\tilde{E}^{ij} = \{\tilde{x}^i, \tilde{x}^j\}$ such that

$$\tilde{E}^{ij} = E^{ij} - 2R^{ij} - R^{im}F_{mk}R^{mj} + O(\tilde{\theta}) = O(\tilde{\theta})\,, \tag{Ap2.10}$$

if coordinates $\{x^i, \theta_j\}$ belong to $\mathcal{M}_{0.1}$ (i.e. $A^i_j = \delta^i_j + O(\theta)$) and matrix R obeys to equation (Ap2.8). Hence map (Ap2.7) obeys condition (Ap2.5).

3. Definition of the map $\mathcal{F}_3$:

$$\mathcal{F}_3(\{x^i, \theta_j\}) = \{\tilde{x}^i, \tilde{\theta}_j\}, \quad \text{where} \quad \tilde{x}^i = x^i - \theta_m \int_0^1 \tau E^{mi}(x, \tau\theta) d\tau, \quad \tilde{\theta}_j = \theta_j\,. \tag{Ap2.11}$$

From (Ap2.1) it follows that transformation (Ap2.11) maps $\mathcal{M}_{r.1}$ in $\mathcal{M}_{r.1}$ if $r \geq 1$. Matrix $E^{ij}(x, \theta)$ transforms to matrix

$$E^{ij} - \frac{2E^{ij}}{r+2} + \frac{1}{r+2}\left(\theta_m \frac{\partial E^{mj}}{\partial \theta_i} + (i \leftrightarrow j)\right) + O(\theta^{r+1})\,. \tag{Ap2.12}$$

On the other hand from Jacoby identity (2.3): $\{x^i\{x^j, x^m\}\} + \{x^j\{x^m, x^i\}\} + \{x^m\{x^i, x^j\}\} = 0$ and from (Ap2.1) it follows that

$$\theta_m \frac{\partial E^{mj}}{\partial \theta_i} + (i \leftrightarrow j) = -\theta_m \frac{\partial E^{ij}}{\partial \theta_m} + O(\theta^{r+1}) = -rE^{ij}(x, \theta) + O(\theta^{r+1})\,. \tag{Ap2.13}$$

Hence (Ap2.12) is equal to zero up to $O(\tilde{\theta}^{r+1})$ and condition (Ap2.5) is obeyed for transformation (Ap2.11).

4. Definition of the map $\mathcal{F}_4$:

$$\mathcal{F}_3(\{x^i, \theta_j\}) = \{\tilde{x}^i, \tilde{\theta}_j\}, \quad \text{where} \quad \tilde{x}^i = x^i,\ \tilde{\theta}_j = \theta_j - \theta_m \int_0^1 P^m_j(x, \tau\theta) d\tau\,. \tag{Ap2.14}$$

We prove that (Ap2.10) maps $\mathcal{M}_{n+1.r}$ in $\mathcal{M}_{n+1.r+1}$ analogously to the proof for (Ap2.11). Suppose that coordinates $\{x^i, \theta_j\}$ belong to $\mathcal{M}_{n+1.r}$ ($r \geq 1$). Then transformation (Ap2.14) maps matrix $P^i_j(x, \theta)$ to matrix

$$P^i_j - \frac{P^i_j}{r+1} + \frac{\theta_m}{r+1}\frac{\partial P^m_j}{\partial \theta_i} + O(\theta^{r+1}) = P^i_j - \frac{P^i_j}{r+1} - \frac{\theta_m}{r+1}\frac{\partial P^i_j}{\partial \theta_m} + O(\theta^{r+1}) = O(\theta^{r+1})\,,$$

because of Jacoby identity $\{x^i, \{x^m, \theta_j\}\} + \{x^m, \{x^i, \theta_j\}\} + \{\theta_j, \{x^i, x^m\}\} = 0$. Hence condition (Ap2.5) is obeyed for transformation (Ap2.14).

Appendix 3. Hamiltonians of adjusted canonical transformations

In this Appendix we prove that for any given adjusted canonical transformation $\{x^i,\theta_j\} \to \{\tilde{x}^i,\tilde{\theta}_j\}$ (2.14a) there exists time-independent Hamiltonian $Q(x,\theta)$ that generates this transformation via differential equations (2.15) and this Hamiltonian is defined uniquely by the condition

$$Q(x,\theta) = Q^{ik}\theta_i\theta_k + \dots\,, \quad \text{i.e. } Q = O(\theta^2)\,. \tag{Ap3.1}$$

For every Hamiltonian (odd function) $Q(x,\theta)$ obeying condition (Ap3.1) consider one-parametric family of functions (Darboux coordinates) $\{y^i(t),\eta_j(t)\}$ $(i,j = 1,\dots,n)$ that are solution to differential equation (2.15):

$$\begin{cases} \frac{dy^i(t)}{dt} = \{Q(y,\eta), y^i\} = -\frac{\partial Q(y,\eta)}{\partial \eta_i}\,, \\ \frac{d\eta_j(t)}{dt} = \{Q(y,\eta), \eta_j\} = \frac{\partial Q(y,\eta)}{\partial y^i}\,, \end{cases} \quad (0 \le t \le 1)\,, \tag{Ap3.2}$$

with initial conditions

$$y^i(t)\big|_{t=0} = x^i,\ \eta_i(t)\big|_{t=0} = \theta_i\,.$$

It is easy to see from explicit expression (2.4) for odd Poisson bracket that if $\{x^i,\theta_j\}$ and $\{\tilde{x}^i,\tilde{\theta}_j\}$ are Darboux coordinates such that $\tilde{x}^i = x^i$ and $\tilde{\theta}_j = O(\theta)$ then $\tilde{\theta}_j = \theta_j$ also. Hence every adjusted canonical transformation $\{x^i,\theta_j\} \to \{\tilde{x}^i,\tilde{\theta}_j\}$ is uniquely defined by functions $\{f^i(x,\theta)\}$ that obey the conditions:

$$\{x^i + f^i(x,\theta), x^j + f^j(x,\theta)\} = 0 \text{ and } \quad f^i(x,\theta) \in O(\theta)\,. \tag{Ap3.3}$$

Statement 3 of Lemma 1 follows from the Lemma:

Lemma 3 *For every set of functions $\{f^i(x,\theta)\}$ $(i = 1,\dots,n)$ obeying conditions* (Ap3.3) *there exists unique Hamiltonian Q obeying condition* (Ap3.1) *such that functions $\{y^i(t)\}$ solutions to differential equation* (Ap3.2) *obey conditions $y^i(t)|_{t=1} = x^i + f^i(x,\theta)$ $(i = 1,\dots,n)$.*

Prove this Lemma.

Consider a ring A of functions on coordinates $(x^1,\dots,x^n,\ \theta_1,\dots,\theta_n)$. (As always functions take values in an arbitrary Grassmann algebra Λ. Consider in A the following gradation: $A_{(p)}$ is a space of functions that are linear combinations of p-th order monoms on variables $\{\theta_1,\dots,\theta_n\}$: $f \in A_{(p)}$ iff $\sum_k \theta_k \frac{\partial f}{\partial \theta_k} = pf$. $A_{(p)} = 0$ for $p \ge n+1$. For every function $f \in A$ we denote by $f_{(p)}$ its component in $A_{(p)}$: $f = f_{(0)} + f_{(1)} + \dots + f_{(n)}$. It is evident that for canonical Poisson bracket (2.4)

$$\{f,g\}_{(p)} = \sum_{i=0}^{n} \{f_{(i)}, g_{(p+1-i)}\}\,. \tag{Ap3.4}$$

Consider also a corresponding filtration:

$$0 = A^{(n+1)} \subset A^{(n)} \subset \ldots \subset A^{(1)} \subset A^{(0)} = A\,,$$

where $A^{(p)} = \oplus_{k\geq p} A_{(k)}$.

We denote by A^+ (A^-) a subspace of even-valued (odd valued) functions in A. Respectively we denote by $A^{\pm}_{(k)} = A_{(k)} \cap A^{\pm}$ and $A^{\pm(k)} = A^{(k)} \cap A^{\pm}$.

We note first that condition (Ap3.1) implies that solutions to equations (Ap3.2) are well defined. Indeed consider arbitrary function $\varphi(x,\theta)$, odd Hamiltonian $Q \in A^{-(2)}$ and differential equation $\dot\varphi = \{Q, \varphi\}$. Projecting this differential equation on the subspace $A_{(p)}$ we come using (Ap3.4) to equations $\dot\varphi_{(p)} = \{Q_{(p+1)}, \varphi_{(0)}\} + \ldots + \{Q_{(2)}, \varphi_{(p-1)}\}$. Function $\varphi_{(0)}$ does not depend on t ($\dot\varphi_0 = 0$) and these equations can be solved recurrently:

$$\varphi_{(p)}\big|_{t=a} = \varphi_{(p)}\big|_{t=0} + a\{Q_{(p+1)}, \varphi_{(0)}\} + \ldots, \qquad \text{(Ap3.5)}$$

where we denote by dots terms depending on $Q_{(2)}, \ldots, Q_{(p)}$ and functions $\varphi_{(0)}, \varphi_{(1)}|_{t=0}, \ldots, \varphi_{(p-1)}|_{t=0}$.

Denote by $\mathcal{N}$ a space of sets of even-valued functions $\{f^i(x,\theta)\}$ ($i = 1, \ldots, n$) such that these functions obey condition (Ap3.3). Consider a map that assigns to every Hamiltonian $Q \in A^{-(2)}$ the solutions $\{y^i(t)|_{t=1}\} = x^i + f^i(x,\theta)$ to differential equations (Ap3.2). Thus we define map $\mathcal{U}$: $A^{-(2)} \to \mathcal{N}$. Relations (Ap3.5) for $\varphi = x^i$ imply that

$$f^i_{(p)} = -\frac{\partial Q_{p+1}}{\partial\theta_i} + \text{terms depending on } Q_{(2)}, \ldots, Q_{(p)}\,. \qquad \text{(Ap3.6)}$$

Consider also a following map δ: $\mathcal{N} \to A^{-(2)}$ such that for every $\{f^i\} \in \mathcal{N}$

$$\delta(\{f^i(x)\}) = -\sum_{i=1,p=1}^{n} \theta_i \frac{f^i_{(p)}(x,\theta)}{p+1} = -\sum_{i=1}^{n} \theta_i \int_0^1 f^i(x, \tau\theta)d\tau\,. \qquad \text{(Ap3.7)}$$

From condition (Ap3.3) for functions $\{f^i\}$ and (2.4) it follows that

$$f^i = -\frac{\partial\tilde Q}{\partial\theta^i} + \sum_m \theta_m \int_{\tau=0}^1 \{f^i, f^m\}|_{x,\tau\theta} d\tau \quad \text{if} \quad \tilde Q = \delta(\{f^i\})\,.$$

Projection of this equation on subspaces $A_{(p)}$ implies

$$f^i_{(p)} = -\frac{\partial\tilde Q_{(p+1)}}{\partial\theta_i} + \text{terms depending on } f^i_{(1)}, \ldots f^i_{(p-1)}. \qquad (Ap3.8)$$

Hence δ is injection. Comparing this relation with relation (Ap3.6) we see that the map $\delta \circ \mathcal{U}$: $A^{-(2)} \to A^{-(2)}$ is bijection. Hence the map $\mathcal{U}$ is also bijection. For every $\{f^i\} \in \mathcal{N}$

the odd function $Q = (\delta \circ \mathcal{U})^{-1} \circ\ \delta(\{f^i\}))$ is the unique Hamiltonian in $A^{(2)}$ required by Lemma.

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