Berezin integral and Berezinian: from identities in the Grothendieck ring of the general linear supergroup to the geometry of Batalin-Vilkovisky quantisation

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Recall definition of Berezin integral

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Integral of function over domain

Coordinates $(x^1, \ldots, x^p; \theta^1, \ldots, x^q)$

$$F(x,\theta) = F_0(x) + F_\alpha(x)\theta^\alpha + F_{\alpha\beta}(x)\theta^\alpha\theta^\beta + \cdots + F_{\mathrm{top}}\theta^1 \ldots \theta^q$$

$$\int F(x,\theta)\mathscr{D}(x,\theta) = \int F_{top}\mathscr{D}(x)$$

$\mathscr{D}(x,\theta) = \mathscr{D}(x^1, \ldots, x^p; \theta^1, \ldots, x^q)$ volume element in superspace.
$\mathscr{D}(x) = dx^1 dx^2 \ldots dx^p$ is volume element in the underlying space.

We suppose that functions on variable x are smooth functions with compact support.

Berezinian—Jacobian of change of coordinates

$$\{x^1,\ldots,x^p;\theta^1,\ldots,\theta^q\}\to\{x^{1'},\ldots,x^{p'};\theta^{1'},\ldots,\theta^{q'}\}$$

$$\begin{cases} x^i = x^i\left(x^{i'},\theta^{\alpha'}\right) \\ \end{cases}$$

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For the Berezin integral $\int F(x,\theta)\mathscr{D}(x,\theta)$:

$$\int F(x,\theta)\mathscr{D}(x,\theta)=\int F(x(x',\theta'),\theta(x',\theta'))\left|\frac{\partial(x,\theta)}{\partial(x',\theta')}\right|\mathscr{D}(x',\theta')$$

$\left|\frac{\partial(x,\theta)}{\partial(x',\theta')}\right|$ Jacobian of change of coordinates, i.e. Berezinian (superdeterminant) of the matrix $\frac{\partial(x,\theta)}{\partial(x',\theta')}$

Berezinian

For $p|q \times p|q$ matrix

$$\frac{\partial(x,\theta)}{\partial(x',\theta')} = \begin{pmatrix} \frac{\partial x(x',\theta')}{\partial x'} & \frac{\partial \theta(x',\theta')}{\partial x'} \\ \frac{\partial x(x',\theta')}{\partial \theta'} & \frac{\partial \theta(x',\theta')}{\partial \theta'} \end{pmatrix} = \begin{pmatrix} M_{00} & M_{10} \\ M_{01} & M_{11} \end{pmatrix}$$

$$\mathrm{Ber}\,\frac{\partial(x,\theta)}{\partial(x',\theta')} = \left|\frac{\partial(x,\theta)}{\partial(x',\theta')}\right| = \frac{\det\left(M_{00} - M_{10} M_{11}^{-1} M_{01}\right)}{\det M_{11}}$$

M_{00}, M_{11} are $p \times p$ and $q \times q$ matrices with even entries
M_{10}, M_{01} are $q \times p$ and $p \times q$ matrices with odd entries

Simple example: no mixing of variables.

Consider changing of variables

$$\{\underbrace{x}_{\text{even}}, \underbrace{\theta, \eta}_{\text{odd}}\} \longrightarrow \{\underbrace{x'}_{\text{even}}, \underbrace{\theta', \eta'}_{\text{odd}}\},$$

Let $x = ax', \theta = b\theta', \eta = c\eta'$. Then

$$\mathrm{Ber}\,\frac{\partial(x,\theta,\varphi)}{\partial(x',\theta,,\varphi')} = \mathrm{Ber}\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} = \frac{a}{bc}$$

If $F(x,\theta) = f(x) + g(x)\theta\varphi$, then

$$\int F(x,\theta,\varphi)\mathscr{D}(x,\theta) = \int (f(x) + g(x)\theta\eta)\,\mathscr{D}(x,\theta,\eta) = \int g(x)dx.$$

$$\int F(x(x',\theta'),\theta(x',\theta'),\varphi'(x',\theta'))\mathrm{Ber}\,\frac{\partial(x,\theta,\varphi)}{\partial(x',\theta',\varphi')}\mathscr{D}(x',\theta') =$$

$$\int (f(ax') + g(ax')bc\theta'\eta')\,\frac{a}{bc}\mathscr{D}(x',\theta',\eta') = \int g(ax')a\mathscr{D}x' = \int g(x)dx\,.$$

Example: mixing variables

Mix even and odd variables

$$\{\underbrace{x}_{\text{even}};\underbrace{\theta,\eta}_{\text{odd}}\}\longrightarrow\{\underbrace{x'}_{\text{even}};\underbrace{\theta',\eta'}_{\text{odd}}\},\qquad \begin{cases} x = x' + b\theta'\eta' \\ \theta = \theta' + cx\theta' \\ \eta = \eta' \end{cases},\qquad a,d>0$$

$$\left|\frac{\partial(x,\theta,\eta)}{\partial(x',\theta',\eta')}\right| = \mathrm{Ber}\begin{pmatrix} 1 & c\theta' & 0 \\ b\eta' & 1+cx & 0 \\ -b\theta' & 1 & 1 \end{pmatrix} = \frac{\det\left(M_{00} - M_{10}M_{11}^{-1}M_{01}\right)}{\det M_{11}} =$$

$$\frac{1-(c\theta',0)\begin{pmatrix}\frac{1}{1+cx'} & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} b\eta' \\ -b\theta'\end{pmatrix}}{(1+c'x)} = \frac{1}{(1+cx')} - \frac{bc\theta'\eta'}{(1+cx')^2}$$

For a function $F(x,\theta,\varphi) = f(x) + g(x)\theta\eta$
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$$\int F(x(x',\theta'),\theta(x',\theta'),\varphi'(x',\theta'))\mathrm{Ber}\,\frac{\partial(x,\theta,\varphi)}{\partial(x',\theta',\varphi')}\mathscr{D}(x',\theta') =$$

$$\int f(x'+b\theta'\eta') + g(x'+b\theta'\eta')(1+cx')\theta'\eta' \times$$

$$\left[\frac{1}{1+cx'} - \frac{bc\theta'\eta'}{(1+cx')^2}\right]\mathscr{D}(x',\theta',\eta') =$$

$$\int \frac{f'(x')b}{1+cx'}dx' - \int \frac{f(x')bc}{(1+cx')^2}dx' + \int g(x')dx' =$$

$$\int \frac{d}{dx}\left(\frac{bf(x)}{1+cx}\right)dx + \int g(x)dx = \int g(x)dx.$$

Characteristic function of linear operator

For a linear operator M on $p|q$-dimensional superspace V consider

$$R_M(z) = \mathrm{Ber}\,(1 + zM)\,.$$

If $M = \mathrm{diag}\,[\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q]$ then

$$R_M(z) = \frac{(1+\lambda_1 z)\dots(1+\lambda_p z)}{(1+\mu_1 z)\dots(1+\mu_q z)} = 1 +$$

$$[(\lambda_1 + \dots + \lambda_p) - (\mu_1 + \dots + \mu_q)]\, z +$$

$$\left[(\lambda_1\lambda_2 + \dots + \lambda_{p-1}\lambda_p) + (\mu_1^2 + \mu_1\mu_2 + \dots + \mu_{q-1}\mu_1 + \mu_q^2)\right] z^2 + \dots$$

Characteristic function. expansion in a vicinity of zero.

One can see that for an arbitrary linear operator on $p|q$-dimensional superspace V

$$R_M(z) = \mathrm{Ber}\,(1 + zA) = \frac{P_M(z)}{Q_M(z)} = \sum_{k=0}^{\infty} c_k(M) z^k\,,$$

where $P_M(z)$ is a polynomial in z of degree $\leq p$, $Q_M(z)$ is a polynomial in z of degree $\leq q$ and

$$c_k(M) = \mathrm{Tr}\wedge^k M, \quad (k = 0, 1, 2, 3, \dots)$$

$\mathrm{Tr}\, A = \mathrm{tr}\, A_{00} - (-1)^{p(A)} \mathrm{tr} A_{11}$.

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In the case if $q = 0$, $R_M(z) = \det(1 + zA)$ is a polynomial of degree p and $c_k(M) = 0$ for $k > p$.

Characteristic function. Expansion in a vicinity of infinity.

We have

$$R_M(z) = \mathrm{Ber}\,(1+z) = z^{p-q}\mathrm{Ber}\,M\,\mathrm{Ber}\,(1+z^{-1}M^{-1}) =$$

$$\mathrm{Ber}\,M \sum_{k=0}^{\infty} z^{p-q-k}\mathrm{Tr}\,\wedge^k M^{-1} =$$

$$\mathrm{Ber}\,M \sum_{k=p-q}^{-\infty} z^k\mathrm{Tr}\,\wedge^{p-q-k} M^{-1}$$

Denote $\mathrm{Ber}\,M\mathrm{Tr}\,\wedge^{p-q-k} M^{-1} = \mathrm{Tr}\,\Sigma^{q+k}M$. It is the trace of representation of M in the space $\Sigma^{q+k}V = \mathrm{Ber}\,V \otimes \wedge^{p-q-k}V^*$.
In pure even case it is just "dual" description.
Now there is a difference.

Two expansions

$$R_M(z) = \mathrm{Ber}\,(1 + zM) =$$

$$= \begin{cases} \sum_{k \geq 0} c_k(M) z^k, & \text{expansion in a vicinity of } z = 0 \\ \sum_{k \leq p-q} c_k^*(M) z^k, & \text{expansion in a vicinity of } z = \infty \end{cases},$$

where

$$\begin{cases} c_k(M) = \mathrm{Tr} \wedge^k M, & (k = 0, 1, 2, 3, \ldots) \\ c_k^*(M) = \mathrm{Tr} \Sigma^{q+k} M = \mathrm{Ber}\, M\, \mathrm{Tr} \wedge^{p-q-k} M^{-1} & (k = p-q, p-q-1, \ldots) \end{cases}$$

Compare series $\{c_k(M)\}$ and $\{c_k^*(M)\}$.
(We assume $c_k(M) = 0$ for $k < 0$ and $c_k^*(M) = 0$ for $k > p - q$.)

Fundamental recurrence relations.

Theorem

For an operator M acting on $p|q$-dimensional vector space the differences

$$\gamma_k(M) = c_k(M) - c_k^*(M) = \mathrm{Tr} \wedge^k M - \mathrm{Tr}\Sigma^{q+k} M$$

form a recurrent sequence with period q (for all $k \in \mathbf{Z}$).

(H.M.K., T.T.Voronov 2005)

$q = 0$. $c_k = c_k^*$, $\mathrm{Tr} \wedge^k M = \det M \,\mathrm{Tr} \wedge^{p-k} M$, $\wedge^k V = \det V \wedge^{p-k} V^*$.
$q = 1$. $\gamma_k = c_k - c_k^*$ form geometric progression: $\gamma_{k+1} = \mu \gamma_k$.
$q = 2$. Then we have $\gamma_{k+2} = \mu \gamma_{k+1} + \nu \gamma_k$. (Fibonacci sequence.)

Hankel determinants.

The conditions that for all k
$\gamma_k(M) = c_k(M) - c_k^*(M)$ form a recurrent sequence with period q is equivalent to the relations

$$\det \begin{pmatrix} \gamma_k(M) & \dots & \gamma_{k+q}(M) \\ \dots & \dots & \dots \\ \gamma_{k+q}(M) & \dots & \gamma_{k+2q}(M) \end{pmatrix} = 0 \qquad (*)$$

for all k.

Fundamental relations for traces

In particular if $k > p - q$ hence $\gamma_k = c_k - c_k^* = c_k$ since $c_k^*(M) = \mathrm{Ber}\, M\, \mathrm{Tr} \wedge^{p-q-k} M^{-1}$ vanish if $k \geq p - q + 1$. We come to relations

$$\det \begin{pmatrix} c_k(M) & \dots & c_{k+q}(M) \\ \dots & \dots & \dots \\ c_{k+q}(M) & \dots & c_{k+2q}(M) \end{pmatrix} = 0 \qquad (**)$$

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for all $k \geq p-q+1$.

Relations $(*)$ and $(**)$ hold for an arbitrary even operator M.

Identities in Grothendieck ring of superspaces

The universal relations formulated above suggest the existence of underlying relations for the spaces themselfs. In particular the relations $(**)$ for traces $c_k(M) = \mathrm{Tr} \wedge^k M$ imply

Theorem

For an arbitrary $p|q$-dimensional vector space V the following identities are obeyed:

$$\det \begin{pmatrix} \wedge^k V & \dots & \wedge^{k+q} V \\ \dots & \dots & \dots \\ \wedge^{k+q} V & \dots & \wedge^{k+2q} V \end{pmatrix} = 0$$

for all $k \geq p - q + 1$.

(H.M.K., T.T.Voronov 2005)

Example of identities for $q = 1$.

If V is $p|1$ dimensional superspace then

$$\det \begin{pmatrix} \wedge^{k} V & \wedge^{k+1} V \\ \wedge^{k+1} V & \wedge^{k+2} V \end{pmatrix} = 0$$

for $k \geq p$, i.e.

$$\wedge^{k} V \otimes \wedge^{k+2} V = \wedge^{k+1} \otimes V \wedge^{k+1} V$$

for all $k \geq p$

General identities in Grothendieck ring of superspaces

The general universal relations $(*)$ for
$\gamma_k(M) = c_k(M) - c_k^*(M) = \mathrm{Tr} \wedge^k M - \mathrm{Tr} \Sigma^{q+k} M$ imply

Theorem

The sequence in the Grothendieck ring

$$\Gamma_k = \wedge^k V - (-\Pi)^q \Sigma^{k+q} V$$

is a recurrent sequence (for all $k \in \mathbf{Z}$).

(H.M.K., T.T.Voronov 2005)

Corollary: Formula for Berezinian

The relations

$$\det \begin{pmatrix} \gamma_k(M) & \dots & \gamma_{k+q}(M) \\ \dots & \dots & \dots \\ \gamma_{k+q}(M) & \dots & \gamma_{k+2q}(M) \end{pmatrix} = 0$$

for $\gamma_k = c_k - c_k^* = \mathrm{Tr} \wedge^k M - \mathrm{Ber}\, M \,\mathrm{Tr} \wedge^{n-k} M$ define all terms $c_k^*(M)$ as rational functions on $\{c_1(M), c_2(M), , c_3(M), \dots$.
In particular

$$\mathrm{Ber}\, M = c^*_{p-q}(M)\,.$$

We arrive at the following Theorem

Formula for Berezinian

Theorem

$$\operatorname{Ber} M = \frac{\det\begin{pmatrix} c_{p-q} & \dots & c_p \\ \dots & \dots & \dots \\ c_p & \dots & c_{p+q} \end{pmatrix}}{\det\begin{pmatrix} c_{p-q+2} & \dots & c_{p+1} \\ \dots & \dots & \dots \\ c_{p+1} & \dots & c_{p+q} \end{pmatrix}}, \qquad c_k = \operatorname{Tr}\wedge^k M\,.$$

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(H.M.K., T.T.Voronov 2005)
Numerator is the trace of representation of operator M in the invariant subspace of the tensors corresponding to rectangular Young diagram with p rows and $q+1$ columns. (Resp. denominator, with $p+1$ rows and q columns).

Example of Berezinian

Example

For $2|2 \times 2|2$ even matrix M

$$\operatorname{Ber} M = \frac{\det \begin{pmatrix} 1 & c_1 & c_2 \\ c_1 & c_2 & c_3 \\ c_2 & c_3 & c_4 \end{pmatrix}}{\det \begin{pmatrix} c_2 & c_3 \\ c_3 & c_4 \end{pmatrix}}, \qquad c_k = \operatorname{Tr} \wedge^k M \,.$$

We recall the basic facts of integration theory over surfaces (submanifolds) in parametric picture (surface is defined by a parameterisation)and in dual one (when surface is defined by equations). We formulate the integration theory using the conception of Berezin integral. Thus we naturally arrive at integration theory over surfaces in superspace. The integration theory formulated in dual picture turns out to be the geometrical basis of Batalin-Vilkovisky prescription of quantisation of general gauge theories.

Differential forms as functions on ΠTN

Let N be a manifold.
Consider tangent bundle TN and the bundle ΠTN reversing parity of coordinates in the fibre.
If $(x^1,\dots,x^n)$—local coordinates in N then $(x^1,\dots,x^n,dx^1,\dots,dx^n)$ local coordinates in ΠTN.
If x^i are even coordinates then dx^i are odd coordinates: $p(dx^i)=p(x^i)+1$.
Functions on ΠTN are differential forms on N:

$$\omega(x,dx)=\underbrace{\omega(x)}_{\text{0-form}}+\underbrace{\omega_i(x)dx^i}_{\text{1-form}}+\dots+\underbrace{\omega_{\rm top}(x)dx^1dx^2\dots dx^n}_{n\text{-form}}$$

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$$\omega(x, dx) = \underbrace{\omega(x)}_{\text{0-form}} + \underbrace{\omega_i(x) dx^i}_{\text{1-form}} + \cdots + \underbrace{\omega_{\text{top}}(x) dx^1 dx^2 \dots dx^n}_{n\text{-form}}$$

If N is a supermanifold then $\omega(x, dx)$ is pseudodifferential form.

Canonical volume form on ΠTN

Let $(x^{i'}, dx^{j'})$ be new coordinates: $\begin{cases} x^i = x^i(x^{i'}) \\ dx^j = \frac{\partial x^j}{\partial x^{j'}} dx^{j'} \end{cases}$. Berezinian of coordinate transformations:

$$\left| \frac{\partial(x, dx)}{\partial(x', dx')} \right| = \mathrm{Ber} \begin{pmatrix} \frac{\partial x}{\partial x'} & \frac{\partial dx}{\partial x'} \\ \frac{\partial x}{\partial dx'} & \frac{\partial dx}{\partial dx'} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial x'} & \frac{\partial^2 x}{\partial x' \partial x'} dx' \\ 0 & \frac{\partial x}{\partial x'} \end{pmatrix} = 1 \,.$$

Thus one can consider canonical volume form $\mathscr{D}(x, dx)$

$$\mathscr{D}(x, dx) = \underbrace{\mathrm{Ber} \frac{\partial(x, dx)}{\partial(x', dx')}}_{\text{equals to } 1} \mathscr{D}(x', dx') \,.$$

and define invariant Berezin integral over ΠTN.

Berezin integral over ΠTN

$$\int_{\Pi TN} \omega(x,dx)\mathscr{D}(x,dx) =$$

$$\int_{\Pi TN} \left(\omega(x) + \omega_i(x)dx^i + \dots + \omega_{\mathrm{top}}(x)dx^1 dx^2 \dots dx^n\right) \mathscr{D}(x,dx) =$$

$$\int_N \omega_{\mathrm{top}}(x)\mathscr{D}(x) = \int_N \omega\,.$$

The integral of a form over N is the Berezin integral over ΠTN with respect to canonical volume form.

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The integral of a form over N is the Berezin integral over ΠTN with respect to canonical volume form.

Thus we arrive at invariant definition of integral of pseudodifferential form $\omega(x, dx)$ in the case if N is a superspace.

Integration over submanifolds (surfaces)

Let C be a surface (submanifold) in N. Let C be defined by a map $D\xrightarrow{\varphi}N$. Let $\omega=\omega(x,dx)$ be a differential form on N. Then $\int_C\omega=\int_D\varphi^*\omega$. In coordinates if $\varphi:\ x^i=x^i(\xi^\alpha)$ then

$$\int_C\omega=\int_D\omega\left(x^i(\xi),\frac{\partial x^i(\xi)}{\partial\xi^\alpha}d\xi^\alpha\right)=$$

$$\int_{\Pi TD}\omega\left(x^i(\xi),\frac{\partial x^i(\xi)}{\partial\xi^\alpha}d\xi^\alpha\right)\mathscr{D}(\xi,d\xi),$$

where $\mathscr{D}(\xi,d\xi)$ is the canonical volume form on the superspace ΠTD of parameters.

$$\int_C\omega=\int_{\Pi TC\subset\Pi TM}\omega\,\mathscr{D}(\xi,d\xi).$$

Parametric and dual picture

One can define a k-dimensional surface C in n-dimensional manifold N by parametric equations $x^i = x^i(\xi^\alpha)$ $(\alpha = 1, \ldots, k)$ or dually by equations $\Psi^a(x) = 0$ $(a = 1, \ldots, n-k)$. In the first case one considers integrals like

$$\int A\left(x(\xi), \frac{\partial x(\xi)}{\partial \xi}\right) \mathscr{D}(\xi).$$

In the dual case one considers integrals like

$$\int \tilde{A}\left(x, \frac{\partial \Psi}{\partial x}\right) \delta(\Psi) \mathscr{D}(x).$$

Example. Flux of a vector field through a surface. Parametric picture

Let C be a surface in $N = \mathbf{E}^3$. Let $\Omega = \rho(x)dx^1dx^2dx^3$ be volume form (differential 3-form) and let $\mathbf{R} = R^i(x)\frac{\partial}{\partial x^i}$ be a vector field on $\mathbf{E}^3$. Consider flux of the vector field $\mathbf{R}$ over the surface C given by parameterisation $x^i = x^i(\xi^\alpha)$

$$\int_C \mathbf{R} d\mathbf{S} =$$

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$$\int_C \mathbf{R}d\mathbf{S} = \int_C \underbrace{\iota_{\mathbf{R}}\Omega}_{\text{2-form}} =$$

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$$\int_C \mathbf{R}d\mathbf{S}=\int_C \underbrace{\iota_{\mathbf{R}}\Omega}_{\text{2-form}} =\int \rho(x^i)\varepsilon_{ikm}R^i(x(\xi))\frac{\partial x^k(\xi)}{\partial \xi^1}\frac{\partial x^m(\xi)}{\partial \xi^2}\mathscr{D}(\xi^1,\xi^2)\,.$$

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$$\int_C \mathbf{R} d\mathbf{S} = \int_C \underbrace{\iota_{\mathbf{R}} \Omega}_{\text{2-form}} = \int \rho(x^i) \varepsilon_{ikm} R^i(x(\xi)) \frac{\partial x^k(\xi)}{\partial \xi^1} \frac{\partial x^m(\xi)}{\partial \xi^2} \mathscr{D}(\xi^1, \xi^2).$$

If $C = \partial D$ is a boundary of the domain then

$$\oint_C \mathbf{R} d\mathbf{S} = \oint_{\partial D} \iota_{\mathbf{R}} \Omega = \int_D d(\iota_{\mathbf{R}} \Omega) = \int_D \mathscr{L}_{\mathbf{R}} \Omega = \int \mathrm{div}\, \mathbf{R} \rho(x) \mathscr{D}(x).$$

(Gauss-Ostrogradsky Theorem)

Example (continued). Flux of a vector field through a surface in the dual picture

Let the surface C be given by a equation $\Psi(x) = 0$.

$$\int_C \mathbf{R} d\mathbf{S} = \int R^i(x) \frac{\partial \Psi(x)}{\partial x^i} \delta(\Psi) \rho(x) \mathscr{D}(x) \, .$$

Parametric picture		Dual picture
$\iota_{\mathbf{R}} \Omega$	$\longleftrightarrow$	$R^i(x) \frac{\partial}{\partial x^i} \rho(x) \mathscr{D}(x)$
2-form		Vector density

Multivector fields as functions on ΠT^*N

ΠT^*N is cotangent bundle to N with reversed parity of fibres.

N—local coordinates (x^i) ,
T^*N—local coordinates (x^i, p_j)
ΠT^*N—-local coordinates (x^i, x^*_i).

If x^i are even then x^*_i are odd. $p(x^*_i) = p(x^i) + 1$.
Functions on ΠT^*N are multivector fields on N:

$$F(x, x^*) = \underbrace{F(x)}_{\text{function}} + \underbrace{F^i(x)x^*_i}_{\text{vector field}} + \underbrace{F^{ij}(x)x^*_i x^*_j}_{\text{bivector field}} + \dots$$

Semidensities on ΠT^*N. A first hint

Let $(x^{i'}, x^*_{j'})$ be new coordinates: $\begin{cases} x^i = x^i(x^{i'}) \\ x^*_j = \frac{\partial x^{j'}}{\partial x^j} x^*_{j'} \end{cases}$.

Berezinian of coordinate transformations: $\left|\frac{\partial(x,x^*)}{\partial(x',x'^*)}\right| =$

$$\mathrm{Ber}\begin{pmatrix} \frac{\partial x}{\partial x'} & \frac{\partial x^*}{\partial x'} \\ \frac{\partial x}{\partial x'^*} & \frac{\partial x^*}{\partial x'^*} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial x'} & \frac{\partial x}{\partial x'}\frac{\partial^2 x'}{\partial x \partial x} x'^* \\ 0 & \frac{\partial x}{\partial x'} \end{pmatrix} = \left(\det\left(\frac{\partial x^i}{\partial x^{i'}}\right)\right)^2 .$$

No canonical volume form, but...

Odd symplectic structure

There is no canonical volume form but there is a canonical odd symplectic structure on $\Pi T^* N$:

$$\omega = dx^i dx_i^* .$$

It generates odd bracket:

$$\{x^i, x_j^*\} = -\{x_j^*, x^i\} = \delta_j^i,\ \{x^i, x^j\} = \{x_i^*, x_j^*\} = 0 . \omega = dx^i dx_i^*$$

$$\{F, G\} = \frac{\partial F}{\partial x^i}\frac{\partial G}{\partial x_i^*} + (-1)^{p(F)}\frac{\partial F}{\partial x_i^*}\frac{\partial G}{\partial x^i} .$$

(Names: odd Poisson bracket, Schouten bracket, Buttin bracket, anti-bracket, BV-bracket).

ΠTN and ΠT^*N

Functions on ΠTN are differential forms on N.

Functions on ΠT^*N are multivector fields on N

Integration objects on N are multivector densities
$= \text{mulitivector fileds} \otimes$ densities.

Warning: Multivector fields are not integration objects.

Example of relations between differential forms and multivector densities (integral forms).

Example

Let ρ be density on N , where N is usual manifold:
$\rho=\rho(x)(x)dx^1dx^2\dots dx^n$ n-form)
Let $\mathbf{F}=F^{i_1\dots i_k}(x)\partial_{i_1}\wedge\partial_{i_k}$ be multivector field on N (i.e. function $F(x,x^*)=F^{i_1\dots i_k}(x)x^*_{i_1}\dots x^*_{i_k}$ on ΠT^*N).
Consider integral form (multivector density) $\mathbf{s}=\mathbf{F}\otimes\rho$.
It defines the $n-k$ form $\omega_{\mathbf{s}}=\iota_{\mathbf{F}}\rho$.
If C is surface of codimension k given by equations $\Psi^a=0$ then

$$\int_C\omega_{\mathbf{F}}=\int_C\iota_{\mathbf{F}}\rho=\int F^{i_1\dots i_k}\frac{\partial\Psi^1}{\partial x^{i_1}}\dots\frac{\partial\Psi^k}{\partial x^{i_k}}\delta(\Psi)\rho(x)\mathscr{D}(x)\,.$$

Differential forms $\overset{\text{Fourier transform}}{\longleftrightarrow}$ Integral forms

Let $\omega(x,dx)$ be function on ΠTN (differential form on N). Consider

$$\omega(x,dx)e^{x_i^* dx^i}\mathscr{D}(x^i,dx^i)$$

Under a change of coordinates exponential $e^{x_i^* dx^i}$ does not change, and $\mathscr{D}(x^i,dx^i)$ is invariant volume form.

$$\underbrace{\omega(x,dx)}_{\text{function on } \Pi TN} \mapsto \underbrace{s(x,x^*)\mathscr{D}(x) = \left[\int \omega(x,dx)e^{x_i^* dx^i}\mathscr{D}(dx)\right]\mathscr{D}(x)}_{\text{function on } \Pi T^*N \otimes \text{ density on } N}$$

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Care must be taken to ensure finitness of integrals over $\mathscr{D}dx$ in the case if N is not purely even manifold. In this case functions on ΠTN represent pseudodifferential forms, and their Fourier transform represent pseudointegral forms.

Differential forms on $N \to$ semidensities on $\Pi T^* N$

Recall that

$$\mathrm{Ber}\begin{pmatrix} \frac{\partial x}{\partial x'} & \frac{\partial x^*}{\partial x'} \\ \frac{\partial x}{\partial x'^*} & \frac{\partial x^*}{\partial x'^*} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial x'} & \frac{\partial x}{\partial x'}\frac{\partial^2 x'}{\partial x \partial x} x'^* \\ 0 & \frac{\partial x}{\partial x'} \end{pmatrix} = \left(\det\left(\frac{\partial x^i}{\partial x^{i'}}\right)\right)^2,$$

i.e. volume form $\mathscr{D}x$ transforms like $\sqrt{\mathscr{D}(x, dx^*)}$. We arrive at the correspondence

$$\underbrace{\omega(x, dx)}_{\text{function on } \Pi TN} \mapsto \underbrace{\mathbf{s}_\omega(x, x^*) = s(x, x^*)\sqrt{\mathscr{D}(x^i, x^*_j)}}_{\text{semidensity on } \Pi T^* N}$$

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From differential forms to semidensities on symplectic supermanifolds

Parametric picture		Dual picture
Diff.forms on N De Rham differential d $C\colon\ x^i = x^i(\xi)$	$\leftrightarrow$	Integral forms on N Divergence operator $C\colon\ \Psi^a(x) = 0$
Function on ΠTN differential $d = dx^i \frac{\partial}{\partial x^i}$ $\Pi TC \subset \Pi TN$	$\to$	Semidensity on ΠT^*N Operator ? ? in ΠT^*N

Canonical odd Laplacian Δ on semidensities in symplectic supermanifold.

Theorem

Let E be an odd symplectic supermanifold.
The expression

$$\Delta \mathbf{s} = \frac{\partial^2 s(x, x^*)}{\partial x^m \partial x^*_m} \sqrt{\mathscr{D}(x, x*)},$$

where $\mathbf{s} = s(x, x^*)\sqrt{\mathscr{D}(x, x*)}$ *is a semidensity in arbitrary Darboux coordinates* [1] *gives well-defined operator on semidensities.* *(H.Kh. 1999)*

---

[1] coordinates $\{x^i, x^*_i\}$ are called Darboux coordinates if odd symplectic structure has canonical appearance in these coordinates: $\{x^i, x^*_j\} = \delta^i_j$, $\{x^i, x^j\} = \{x^*_i, x^*_j\} = 0$.

Batalin-Vilkovisky identity

Let (x^i, x^*_j) and $(x^{i'}, x^*_{j'})$ be a pair of two arbitrary Darboux coordinates. Consider $\mathbf{s} = \sqrt{\mathscr{D}(\mathbf{x}, \mathbf{x}^*)}$. Evidently $\Delta\mathbf{s} = 0$. On the other hand

$$\mathbf{s} = \sqrt{\mathscr{D}(x, x^*)} = \sqrt{\mathrm{Ber}\,\frac{\partial(x, x^*)}{\partial(x', x'^*)}}\sqrt{\mathscr{D}(x, x^*)}\,.$$

Hence calculating $\Delta\mathbf{s}$ in new Darboux coordinates we come to Batalin-Vilkovisky identity

$$\frac{\partial^2}{\partial x^m \partial x^*_m}\sqrt{\mathrm{Ber}\,\frac{\partial(x, x^*)}{\partial(x', x'^*)}} = 0\,.$$

This highly non-trivial identity (I.Batalin, G.Vilkovisky , 1981) is the cornerstone of BV geometry. The construction of odd canonical Laplacian illuminates the geometrical meaning of Batalin-Vilkovisky identity.

From de Rham differential on N to Δ-operator on $\Pi T^* N$

We know that $\Pi T^* N$ has canonical odd symplectic structure. Consider

$$\omega(x,dx) \mapsto \mathbf{s}_\omega = \left(\int \omega(x,dx) e^{x^*_m dx^m} \mathscr{D}(dx)\right) \sqrt{\mathscr{D}(x,x^*)}$$

Then

$$\mathbf{s}_{d\omega} = \Delta(\mathbf{s}_\omega)\,.$$

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Then

$$\mathbf{s}_{d\omega} = \Delta(\mathbf{s}_\omega)\,.$$

Remark The group of symplectomorphisms of $\Pi T^* N$ contains the group of diffeomorphisms of N and it is "bigger". To diffeomorphism $x^i = x^i(x^{i'})$ corresponds symplectomorphism $x^i = x^i(x^{i'}), x^*_j = \frac{\partial x^{j'}}{\partial x^j} x^*_{j'}$. One can also consider sympelctomorphism which destroys fiber bundle of $\Pi T^* N$.

Integration over surface in Berezin integral approach

Let C be a surface in N and ω be a differential form. We know

$$\int_C \omega = \int \omega(x, dx))\big|_{x=x(\xi, dx=\frac{\partial x(\xi)}{\partial \xi} d\xi)} \mathscr{D}(\xi, d\xi)\,.$$

Taking Fourier transform we come to dual picture

$$\int_C \omega = \int s_\omega\left(x^i, x_j^*\right)\big|_{x_i^*=\frac{\partial \Psi^a(x)}{\partial x^i}\eta_a} \prod_b \delta(\Psi^b) \mathscr{D}(\eta)\mathscr{D}(x),$$

where equations $\Psi^a(x) = 0$ define the surface C, η^a are odd variables and integral form $\Sigma_\omega(x, x^*)\mathscr{D}(x)$ is the Fourier transform of differential form ω:

$$s_\omega(x, x^*)\mathscr{D}(x) = \left[\int \omega(x, dx) e^{x_i^* dx^i} \mathscr{D}(dx)\right] \mathscr{D}(x)\,.$$

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$$s_\omega(x,x^*)\mathscr{D}(x) = \left[\int \omega(x,dx)e^{x_i^* dx^i}\mathscr{D}(dx)\right]\mathscr{D}(x)\,.$$

Notice that under $\Psi^a \mapsto \Psi^{a'} = \lambda_a^{a'}\Psi^a$ $\Sigma|_C$ is multiplying on $\det\lambda$ and $\prod_b \delta(\Psi^b)$ is dividing on $\det\lambda$. Integral remains unchanged.

Integration over $C \to$ Integration of semidensity over Lagrangian surface in ΠT^*N

We rewrite the former integral

$$\int_C \omega = \int s_\omega \left(x^i, x^*_i\right) \prod_{i,b} \delta\left(x^*_i - \frac{\partial \Psi^a(x)}{\partial x^i}\eta_a\right) \delta(\Psi^b)\mathscr{D}(\eta)\mathscr{D}(x,x^*)$$

One can see that submanifold specified by equations

$$\Lambda_C = \left\{(x,x^*)\colon\ x^*_i - \frac{\partial \Psi^a(x)}{\partial x^i}\eta_a = 0, \Psi^a(x) = 0\right\}$$

is Lagrangian surface in ΠT^*N:

$$dx^i dx^*_i|_{\Lambda_C} = dx^i d\left(\frac{\partial \Psi^a(x)}{\partial x^i}\eta_a\right) = dx^i dx^j \frac{\partial^2 \Psi}{\partial x^i \partial x^j}\eta^a + d\Psi^a d\eta^a = 0\,.$$

Integration over surface C is reduced to the integration of semidensity $\mathbf{s}_\omega$ over the Lagrangian surface Λ_C.

There is a canonical construction in odd symplecctic geometry which allows one to integrate semidensity over an arbitrary Lagrangian surface. (A.S.Schwarz 1993, A.P.Nersessian, H.M.Kh. 1995). The former integral is in fact just manifestation of this picture.

From differential forms to semidensities in symplectic supermanifolds. (Revisited)

Parametric picture		Dual picture
Diff.form on N De Rham differential d $C\colon\ x^i = x^i(\xi)$	$\leftrightarrow$	Multivector dens. on N Divergence operator $C\colon\ \Psi^a(x) = 0$
Function on ΠTN differential $d = dx^i \frac{\partial}{\partial x^i}$ $\Pi TC \subset \Pi TN$	$\to$	Semidensity on ΠT^*N Canonical odd Laplacian Δ Lagr. sur.$\Lambda_C \subset \Pi TN$

$$C\colon\ \Psi^a = 0,\ \Lambda_C = \left\{(x, x^*)\colon\ x^*_i = \frac{\partial \Psi^a(x)}{\partial x^i}\eta_a, \Psi^a(x) = 0\right\}$$

$$\int_C \omega = \int_{\Lambda_C} \mathbf{s}_\omega, \quad \mathbf{s}_{d\omega} = \Delta\left(\mathbf{s}_\omega\right).$$

Batalin-Vilkovisky geometry

$S(\varphi)$—action of theory
$\{\mathbf{R}_\alpha\}$ symmetries: $R^i_\alpha \frac{\delta S}{\delta \varphi^i} = R^i_\alpha \mathscr{F}_i = 0$.

$$[\mathbf{R}_\alpha, \mathbf{R}_\beta] = t^\gamma_{\alpha\beta} \mathbf{R}_\gamma$$

Batalin-Vilkovisky geometry

$S(\varphi)$—action of theory
$\{\mathbf{R}_\alpha\}$ symmetries: $R^i_\alpha \frac{\delta S}{\delta \varphi^i} = R^i_\alpha \mathscr{F}_i = 0$.

$$[\mathbf{R}_\alpha, \mathbf{R}_\beta] = t^\gamma_{\alpha\beta} \mathbf{R}_\gamma + T^{ij}_{\alpha\mathbf{b}} \mathscr{F}_j \frac{\delta}{\delta_j}$$

$T \neq 0$, open algebra of symmetries
To C: $\Psi^a(\varphi) = 0$ (surface of gauge conditions in the space of fields) corresponds the Lagrangian surface Λ_C in the symplectic space of fields and antifields. Fields—$\Phi^A = (\varphi^\alpha, c_\alpha)$
Antifields—-$\Phi^*_A = (\varphi^*_\alpha, c^*_\alpha)$
Odd symplectic superspace is the space of fields and antifields with canonical symplectic structure.

The partition function is the integral of semidensity $e^{\mathscr{S}}\sqrt{\mathscr{D}(\Phi,\Phi^*}$ over Lagrangian surface, where master-action $\mathscr{S}$ is defined by the initial conditions

$$\mathscr{S} = S(\varphi) + c_\alpha R^i_\alpha \varphi^*_i + \dots$$

and BV-equation

$$\Delta \mathbf{s} = 0\,.$$

If algebra of symmetries is abelian: $t_{\alpha\beta\gamma} = T^{ij}_{\alpha\beta} = 0$ then $\mathscr{S} = S(\varphi) + c_\alpha R^i_\alpha \varphi^*_i$.
If it is closed Lie algebra: $t^\gamma_{\alpha\beta}$ are constants and $T^{ij}_{\alpha\beta} = 0$ then $\mathscr{S} = S(\varphi) + c_\alpha R^i_\alpha \varphi^*_i + t^\gamma_{\alpha\mathbf{b}} c_\alpha c_\beta c^*_\gamma$.

The transformation of symmetries

$$R^i_\alpha \mapsto \lambda^\beta_\alpha R^i_\beta + E^{[ij]}_\alpha \mathscr{F}_j$$

can be coded by corresponding canonical transformation in the odd symplectic superspace of fields and antifields.
Gauge independence = The integral of semidensity does not change under variation of Lagrangian surface.

$$\left(\int_{C+\delta C} \omega = \int_C \quad \text{if} \qquad d\omega = 0\right) \to \left(\int_{\Lambda+\delta\Lambda} \mathbf{s} = \int_\Lambda \mathbf{s} \quad \text{if} \qquad \Delta \mathbf{s} = 0\right)$$

(A.S. Schwarz 1993, H.M.K. 1999)

Second order operator on the algebra of densities

Contravariant tensor S^{ab}, upper connection γ^a	$\longleftrightarrow$	Second order self-adjoint operator on algebra of densities

(H.Kh., T.Voronov 2003)

$$\Delta a(x,t) = \Delta^+ a(x,t) =$$

$$\frac{1}{2}\left(\partial_a S^{ab}\partial_b + (2\hat\sigma - 1)\gamma^a\partial_a + \hat\sigma\partial_a\gamma^a + \hat\sigma(\hat\sigma - 1)\theta\right) a(x,t)\,.$$

Here $\sigma = t\frac{d}{dt}$ is operator of the weight of density, and $\theta = \gamma^a S_{ab}\gamma^b$. In the case if S^{ab} is invertible then $\gamma^a = S^{ab}\gamma_b$, where γ_b is a connection on volume forms.
One can consider $\gamma_a = -\Gamma^b_{ba}$, where Γ^b_{ba} are Christoffel of affine connection.

Canonical pencil of operators

Restricting the operator Δ on densities of weight σ we arrive at the operator pencil Δ_σ,

$$\Delta_\sigma\left(a(x)|Dx|^\sigma\right) =$$

$$\frac{1}{2}\left(\partial_a S^{ab}\partial_b + (2\sigma-1)\gamma^a\partial_a + \sigma\partial_a\gamma^a + \sigma(\sigma-1)\theta\right)a(x)|Dx|^\sigma\,,$$

$\sigma \in \mathbf{R}$.

Special case: operators on semidensities, $\sigma = \frac{1}{2}$.

Fix S^{ab}. Choose an arbitrary connection γ_a. Consider the canonical pencil at $\sigma = \frac{1}{2}$.

$$\Delta^{\gamma}_{\frac{1}{2}}\left(a(x)\sqrt{|Dx|}\right) = \frac{1}{2}\left(\partial_a\left(S^{ab}\partial_b a(x)\right) + \frac{\partial_a\gamma^a}{2}a(x) - \frac{\gamma^a\gamma_a}{4}a(x)\right)\sqrt{|Dx$$

How this operator changes if we change the connection γ?

$$\gamma \to \gamma' = \gamma + \mathbf{X}, \quad \Delta^{\gamma}_{\frac{1}{2}} \to \Delta^{\gamma'}_{\frac{1}{2}} = \Delta^{\gamma}_{\frac{1}{2}} + \frac{1}{4}\partial_a X^a - \frac{1}{8}\left(2\gamma_a X^a + X_a X^a\right) =$$

$$\Delta^{\gamma}_{\frac{1}{2}} + \frac{1}{4}\left(\partial_a X^a - \gamma_a X^a\right) - \frac{1}{8}\mathbf{X}^2 = \Delta^{\gamma}_{\frac{1}{2}} + \frac{1}{4}\left(\operatorname{div}_\gamma \mathbf{X} - \frac{1}{2}\mathbf{X}^2\right).$$

$$\Delta^{\gamma}_{\frac{1}{2}} = \Delta^{\gamma'}_{\frac{1}{2}} \qquad \Leftrightarrow \qquad \operatorname{div}_\gamma \mathbf{X} - \frac{1}{2}\mathbf{X}^2 = 0.$$

Groupoid of connections

Let A be an affine space of all connections on volume forms.
Arrow: $\gamma \xrightarrow{\mathbf{X}} \gamma'$ such that $\gamma, \gamma' \in A$ and $\gamma' = \gamma + \mathbf{X}$.

Set S of admissible arrows: $S = \{\gamma \xrightarrow{\mathbf{X}} \gamma' : \quad \mathrm{div}_{\gamma} \mathbf{X} - \frac{1}{2}\mathbf{X}^2 = 0\}$

Inverse arrow: If $\gamma \xrightarrow{\mathbf{X}} \gamma' \in S$ then $\gamma' \xrightarrow{-\mathbf{X}} \gamma \in S$.
(If $\mathrm{div}_{\gamma}\mathbf{X} - \frac{1}{2}\mathbf{X}^2 = 0$ then $-\mathrm{div}_{\gamma+\mathbf{X}}\mathbf{X} - \frac{1}{2}\mathbf{X}^2 = 0$).

Multiplication of arrows: if $\gamma_1 \xrightarrow{\mathbf{X}} \gamma_2,\ \gamma_2 \xrightarrow{\mathbf{Y}} \gamma_3 \in S$ then $\gamma_1 \xrightarrow{\mathbf{X}+\mathbf{Y}} \gamma_3 \in S$.

$$(\text{if } \mathrm{div}_{\gamma_1}\mathbf{X} - \frac{1}{2}\mathbf{X}^2 = \mathrm{div}_{\gamma_2}\mathbf{Y} - \frac{1}{2}\mathbf{Y}^2 = 0 \text{ then } \mathrm{div}_{\gamma_1}(\mathbf{X}+\mathbf{Y}) - \frac{1}{2}(\mathbf{X}+\mathbf{Y})^2 = 0\,.)$$

We call this groupoid the Batalin-Vilkovisky groupoid.
(H.Kh., T. Voronov.)

Conclusion

Operator $\Delta^{\gamma}_{\frac{1}{2}}$ depends not on a connection but only on its equivalence class, the groupoid orbit $\mathcal{O}_\gamma$ of a connection γ,

$$\mathcal{O}_\gamma = \{\gamma' : \quad \gamma \xrightarrow{\mathbf{X}} \gamma' \in S\}.$$

$$\Delta^{\gamma}_{\frac{1}{2}} = \Delta^{\gamma'}_{\frac{1}{2}} \qquad \Leftrightarrow \qquad \mathrm{div}_{\gamma}\mathbf{X} - \frac{1}{2}\mathbf{X}^2 = 0\,.$$

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Where such operators naturally arise?

Consider an odd symplectic supermanifold M with arbitrary Darboux coordinates $z^A = \{x^i, x^*_j\}$
There is no canonical volume form (no Liouville Theorem!) and no canonical connection on volume forms.
There are many affine connections compatible with the symplectic structure. One cannot choose a unique "Levi-Civita" connection Γ^A_{BC}.

One cannot choose a distinguished connection on volume forms.

Can we choose a class of connections?

Canonical class of connections

Definition

We say that γ_A is a Darboux flat connection if there exist Darboux coordinates such that $\gamma_A \equiv 0$ in these Darboux coordinates.

Theorem

All Darboux flat connections belong to the same orbit of the Batalin-Vilkovisky groupoid. That means that for two Darboux flat connections γ_1, γ_2

$$\gamma_1 \xrightarrow{\mathbf{X}} \gamma_2 \in S, \text{ i.e. } \operatorname{div} \mathbf{X} - \frac{1}{2}\mathbf{X}^2 = 0\,,$$

(I.A.Batalin, G.A.Vilkovisky [2] *—H.Kh.—H.Kh.,T.Voronov)*

[2] The statement relies on the Batalin-Vilkovisky identity

Canonical Δ-operator on semidensitites revisited

Let γ be an arbitrary Darboux flat connection and $\{z^A\}$ be arbitrary Darboux coordinates. Then

$$\Delta^{\mathscr{O}_\gamma}_{\frac{1}{2}}\left(a(x,x^*)\sqrt{\mathscr{D}(x,x*)}\right) =$$

Canonical Δ-operator on semidensitites revisited

Let γ be an arbitrary Darboux flat connection and $\{z^A\}$ be arbitrary Darboux coordinates. Then

$$\Delta^{\mathscr{O}_\gamma}_{\frac{1}{2}}\left(a(x,x^*)\sqrt{\mathscr{D}(x,x*)}\right) =$$

$$\frac{\partial^2 a(x,x^*)}{\partial x^i \partial x^*_i} + \frac{\partial_A \gamma^A}{2} a(x,x*) - \frac{\gamma^A \gamma_A}{4} a(x,x*)\sqrt{\mathscr{D}(x,x*)}$$

Canonical Δ-operator on semidensitites revisited

Let γ be an arbitrary Darboux flat connection and $\{z^A\}$ be arbitrary Darboux coordinates. Then

$$\Delta_{\frac{1}{2}}^{\mathcal{O}_\gamma}\left(a(x,x^*)\sqrt{\mathcal{D}(x,x*)}\right) =$$

$$\frac{\partial^2 a(x,x^*)}{\partial x^i \partial x_i^*} + \frac{\partial_A \gamma^A}{2} a(x,x*) - \frac{\gamma^A \gamma_A}{4} a(x,x*)\sqrt{\mathcal{D}(x,x*)}$$

$$= \Delta a.$$

according to Theorem above, since $\frac{\partial_A \gamma^A}{2} - \frac{\gamma^A \gamma_A}{4} = 0$ for an arbitrary Darboux flat connection.

Scalar curvature of connection compatible with volume form

For an arbitrary volume form one ρ one can consider a scalar function

$$\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\,.$$

One very interesting observation:
This scalar function equals (up to a coefficient) to the scalar curvature of an arbitrary affine connecction which is compatible with symplectic structure and the volume form ρ.
(I.Batalin, K.Bering 2007)

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