Geometry of Differential operators

H.M. Khudaverdian
The University of Manchester. School of Mathematics.

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This is summary of lectures which I had on the School at Workshop. Lectures contain textbook staff+ something that I did with Ted Voronov (mostly the first part) and something that I did with Adam Biggs (end of the second part)). I also was inspired by very good book [8].

Contents

1 Differential operators and algebra of densities

1.1 Differential operators on functions

M is a manifold.

We say that Δ is an operator on functions on M of the order $\leq n$ if for an arbitrary function g $\Delta_g = g \circ \Delta - \Delta \circ g$ is an operator of the order $\leq n-1$. (Non-zero operator L has order 0 if it is linear with respect to algebra of functions:$L(fg) = fL(g)$.)

Every linear operator that obeys the identity

$$\mathbf{X}(fg) = f\mathbf{X}(g) + g\mathbf{X}(f)$$

is a vector field. One can see that for first order operator L, $L = \mathbf{X} + R$, $R = L(1)$ is a function (scalar) and $\mathbf{X} = L - R$ is a vector field. What about higher order operators?

If Δ is n-th order operator on functions then

$$\Delta = S^{i_1 i_2 \dots i_n} \partial_{i_1} \dots \partial_{i_n} + T^{i_1 i_2 \dots i_{n-1}} \partial_{i_1} \dots \partial_{i_{n-1}} + \dots$$

The first term $S^{i_1 i_2 \dots i_n} \partial_{i_1} \dots \partial_{i_n}$ transforms in the following way under changing of coordinates:

$$S^{i_1 i_2 \dots i_n} = \frac{\partial x^{i'_1}}{\partial x^{i_1}} \frac{\partial x^{i'_2}}{\partial x^{i_2}} \dots \frac{\partial x^{i'_n}}{\partial x^{i_n}} S^{i_1 i_2 \dots i_n}$$

This is symmetric n-th order contravariant tensor i *principal symbol* of operator Δ.

What about the next terms?

Consider this question in more detail for second order operators

1.1.1 Second order operators on functions and connections

Let Δ be second order operator. In local coordinates $\Delta = S^{ij}\partial_i\partial_j + T^i\partial_i$.

We already know that first order operator is vector field+scalar. Use it. Consider a scalar product $\langle\ ,\ \rangle_{\boldsymbol{\rho}} = \int_M fg\boldsymbol{\rho}$, where $\boldsymbol{\rho} = \rho(x)|Dx|$ is an arbitrary volume form. Then consider adjoint operator with respect to this scalar product :

$$\Delta^+ = \frac{1}{\rho}\partial_i\partial_j S^{ij} - \frac{1}{\rho}\partial_i T^i + R\,.$$

We see that

$$\Delta^+ - \Delta = 2\partial_r S^{ri} + 2S^{ik}\partial_k \log\rho\partial_i - 2T^i\partial_i + \dots$$

is first order operator. Hence

$$K^i = 2\partial_r S^{ri} + 2S^{ik}\partial_k \log\rho\partial_i - 2T^i\partial_i$$

is the vector field. The term $\gamma_k = -\partial_k \log\rho$ defines connection...

What is it a connection? It defines derivative of an arbitrary volume form:

$$\nabla_{\mathbf{X}}(f\boldsymbol{\rho}') = \partial_{\mathbf{X}} f\boldsymbol{\rho}' + f\nabla_{\mathbf{X}}\boldsymbol{\rho}'$$

Denote by γ_k: $\quad \gamma_k = \nabla_k|Dx|$ in given coordinates. Then

$$\nabla_{\mathbf{X}}(f\boldsymbol{\rho}') = \nabla_{\mathbf{X}}(f\rho'(x)|Dx|) = X^i\left(\partial_i(f\rho') + \gamma_i f\rho'(x)\right)|Dx|\,.$$

An arbitrary volume $\boldsymbol{\rho}$ form defines flat connection ∇:

$$\nabla_{\mathbf{X}}\colon\ \nabla_{\boldsymbol{\rho}}(\boldsymbol{\rho}') = \partial_{\mathbf{X}}\left(\frac{\boldsymbol{\rho}'}{\boldsymbol{\rho}}\right)$$

with $\Gamma_k = -\partial_k \log\boldsymbol{\rho}$.

Returning to our second order operator we see that

$$T^i = \partial_r S^{ri} + S^{ik}\partial_k \log\rho - \frac{1}{2}K^i = \partial_r S^{ri} + \Gamma^i - \frac{1}{2}K^i = \partial_r S^{ri} + \gamma^i$$

Fact In the second order operator $\Delta = S^{ij}\partial_i\partial_j + T^i\partial_i + R$, the combination $T^i - \partial_r S^{ri}$ is *upper connection* or in the other words: An arbitrary second order operator Δ has an appearance:

$$\Delta f = \partial_i\left(S^{ik}\partial_k f\right) - \gamma^i\partial_i + R$$

where γ^i upper connection, R scalar. We see that in second order operators connection appears....

Example In Riemannian manifold one can consider connection $\gamma_i = -\Gamma^k_{ik}$. This connection with Riemannian metrics defines the well-known Beltrami-Laplace operator:

$$\Delta_{B.L} f = \partial_i \left(g^{ik \partial_k f} \right) - \gamma^i \partial_i \,. \tag{1.1}$$

1.2 Algebra of densities.

We will go to densities. (This is very useful.) Density $\boldsymbol{s} = s(x)|Dx|^\lambda$ of the weight λ. Under changing of coordinates it is multiplied on λ-th power of Jacobian. (M is orientable manifold with chosen class of orientation.)

Examples of densities

- Functions –densities of weight $\lambda = 0$.
- Volume forms–densities of the weight $\lambda = 1$
- Wave function–densities of the weight $\lambda = \frac{1}{2}$
- Schwarzian of diffeomorphism—densities of the weight $\lambda = 2$...

1.2.1 Lie derivative of densities

In local coordinates

$$\mathcal{L}_{\mathbf{X}}(\boldsymbol{s}) = \left(X^i \partial_i s(x) + \lambda \partial_i X^i s(x) \right) |Dx|$$

One can easy check its invariance. If $\lambda = 1$ we come to divergence

$$\operatorname{div}_{\boldsymbol{\rho}} \mathbf{X} = \frac{1}{\boldsymbol{\rho}} \mathcal{L}_{\mathbf{X}}(\boldsymbol{\rho}) = \left(X^i \partial_i \rho(x) + \lambda \partial_i X^i \rho(x) \right) |Dx| = \frac{1}{\rho(x)} \partial_1 \left(\rho(x) X^i \right)$$

Exercise In Riemannian case where $\rho = \sqrt{\det g}$. Compare with covariant divergence.

Exercise For Beltrami Laplace operator

$$\Delta_{B.L} f = \operatorname{div}_{\boldsymbol{\rho}} \mathrm{gradf} = \frac{1}{\rho(x)} \partial_1 \left(\rho(x) g^{ik} \partial_k f \right)$$

1.2.2 Algebra of densities. Scalar product. Extended manifold $\widehat{M}$.

Density on M is a polynomial function on $\widehat{M}$.

Local coordinates on $\widehat{M}$ are (x^i, t), $t \approx |Dx|$. Globally defined Euler operator:

$$\widehat{\lambda} = t\frac{\partial}{\partial t}$$

Fact *This is first order operator, vector field on $\widehat{M}$.*

On the algebra $\mathcal{F}(M)$ of densities one can consider the canonical scalar product $\langle\ ,\ \rangle$ defined by the following formula: if $\boldsymbol{s}_1 = s_1(x)\mathcal{D}x^{\lambda_1}$ and $\boldsymbol{s}_2 = s_2(x)\mathcal{D}x^{\lambda_2}$ then

$$\langle \boldsymbol{s}_1, \boldsymbol{s}_2\rangle = \begin{cases} \int_M s_1(x)s_2(x)\mathcal{D}x\,, & \text{if } \lambda_1+\lambda_2 = 1\,, \\ \\ 0 \text{ if} & \text{if } \lambda_1+\lambda_2 \neq 1\,. \end{cases} \tag{1.2}$$

One can see that

$$x^{i+} = x^i, \quad \left(\frac{\partial}{\partial x^i}\right)^+ = -\left(\frac{\partial}{\partial x^i}\right), \quad t^+ = t, \quad \left(\frac{\partial}{\partial t}\right)^+ = \frac{2}{t}-\left(\frac{\partial}{\partial t}\right) \quad \text{and} \quad \widehat{\lambda}^+ = 1-\widehat{\lambda}\,,$$

where L^+ is adjoint to L with respect to the canonical scalar product: $\langle L\boldsymbol{s}_1, \boldsymbol{s}_2\rangle = \langle \boldsymbol{s}_1, L^+\boldsymbol{s}_2\rangle$. (We suppose that M is compact orientable manifold and only orientation preserving coordinate transformations are allowed. See for details [6]).

1.2.3 Conjugate operators. Vector fields on $\widehat{M}$

Vector fields of weight δ:

$$\widehat{\mathbf{X}} = t^\delta\left(X^i\partial_i + X^0\widehat{\lambda}\right)$$

Definition-Proposition

$$\operatorname{div}\mathbf{X} = -\left(\widehat{\mathbf{X}}^+ + \widehat{\mathbf{X}}\right) = t^\delta\left(\partial_i X^i\partial_i + (\delta-1))X^0\right).$$

Fact: Lie derivative—-divergence less vector field:

$$\widehat{\mathbf{X}} = X^i\partial_i - \widehat{\lambda}\partial_i\mathbf{X}^i$$

Exercise A connection on M defines lifting

$$\mathbf{X} \mapsto \widehat{\mathbf{X}}_\gamma = X^i \partial_i + \gamma_i X^i \widehat{\lambda}$$

Thus we come to

$$\operatorname{div}_\gamma \mathbf{X} = \operatorname{div} \widehat{\mathbf{X}}_\gamma$$

Remark The divergence possesses curvature:

$$\operatorname{div}_\gamma [\mathbf{X}, \mathbf{Y}] = \partial_{\mathbf{X}} \operatorname{div}_\gamma \mathbf{Y} - \partial_{\mathbf{Y}} \operatorname{div}_\gamma \mathbf{X} + \mathcal{F}(\mathbf{X}, \mathbf{Y})$$

where $\mathcal{F}_{ik} = \partial_i \gamma_k - \partial_k \gamma_i$ is a curvature of connection. (Recall that Ricci tensor for general affine connection is not symmetric.)

Exercise: An arbitrary vector field $\widehat{\mathbf{X}}$ is a sum of Lie derivative and vertical vector fields.

One can consider another lifting $\mathbf{X} \mapsto \mathcal{L}_{\mathbf{X}}$...... Difference of these two liftings is $\widehat{\lambda} \operatorname{div}_\gamma \mathbf{X}$...

(See more in details about connection and vector fields on extended manifolds in [6] and [7]).

1.3 Second order operators on $\widehat{M}$

Operator pencil $\{\Delta_\lambda\}$—operator on $\widehat{M}$ which is polynomial on λ).

Consider useful example. Let $\boldsymbol{\rho}$ be an arbitrary volume form and S^{ik} second order principal symbol. One can define the pencil of operators:

$$\Delta_\lambda \colon \Delta_\lambda \sigma = \boldsymbol{\rho}^{\lambda-1} \partial_i \left[\boldsymbol{\rho} S^{ik} \partial_k \left(\frac{\boldsymbol{s}}{\boldsymbol{\rho}^\lambda} \right) \right] =$$

$$S^{ik} \partial_i \partial_k + \left(\partial_r S^{ri} + (2\lambda - 1)\Gamma^i \right) \partial_i + \left(\lambda \partial_i \Gamma^i + \lambda(\lambda - 1)\Gamma^i \Gamma_i \right) \tag{1.3}$$

where $\Gamma_i = -\partial_i \log \rho(x)$ is flat connection defined by the volume form.

One can see that this is self-adjoint operator:

$$\Delta_\lambda^+ = \Delta_{1-\lambda} \,.$$

In terms of operator on extended manifold $\widehat{M}$ ($\lambda \mapsto \widehat{\lambda}$):

$$\widehat{\Delta} = \widehat{\Delta}^+ = S^{ik} \partial_i \partial_k + \left(\partial_r S^{ri} + (2\widehat{\lambda} - 1)\Gamma^i \right) \partial_i + \left(\widehat{\lambda} \partial_i \Gamma^i + \widehat{\lambda}(\widehat{\lambda} - 1)\Gamma^i \Gamma_i \right) \,.$$

Remark Note that the operator $(2\widehat{\lambda}-1)\mathcal{L}_{\mathbf{X}}$ is second order self-adjoint operator on M. It corresponds to the pencil $L_\lambda = (2\lambda-1)\mathcal{L}_{\mathbf{X}}$ of first order operators.

Consider the operator: $\widehat{\Delta} + (2\widehat{\lambda}-1)\mathcal{L}_{\mathbf{X}} =$

$$S^{ik}\partial_i\partial_k + \left(\partial_r S^{ri} + (2\lambda-1)\Gamma^i\right)\partial_i + \left(\lambda\partial_i\Gamma^i + \lambda(\lambda-1)\Gamma^i\Gamma_i\right) + (2\widehat{\lambda}-1)X^i\partial_i + \widehat{\lambda}(2\widehat{\lambda}-1)\partial_i X^i =$$

$$S^{ik}\partial_i\partial_k + \left(\partial_r S^{ri} + (2\lambda-1)\gamma^i\right)\partial_i + \left(\lambda\partial_i\gamma^i + \lambda(\lambda-1)\theta\right) . \tag{1.4}$$

where

$$\gamma^i = \Gamma^i + X^i, \qquad \theta = (\Gamma_i\Gamma^i + 2\Gamma_i X^i) + 2\mathrm{div}_{\,\Gamma}\mathbf{X}$$

In the case if symbol is invertible

$$\theta = (\Gamma_i\Gamma^i + 2\Gamma_i X^i) + 2\mathrm{div}_{\,\Gamma}\mathbf{X} = \gamma_i\gamma^i + 2\mathrm{div}_{\,\Gamma}\mathbf{X} - \mathbf{X}^2 .$$

This is a basic example. Namely

Theorem (Vor. Kh.2003) Let $\Delta \in \mathcal{D}^{(2)}_{\lambda_0}(M)$ be second order operator defined on densities of weight λ_0. In the case if $\lambda_0 \neq 0, 1, \frac{1}{2}$ there exists unique operator pencil Δ_λ of the order 2, i.e. the second order operator $\widehat{\Delta}$ on $\widehat{M}$ such that

- $\widehat{\Delta}\big|_{\lambda=\lambda_0} = \Delta_{\lambda_0}$.
- $\widehat{\Delta} = \widehat{\Delta}^+$, i.e. $\Delta^+_\lambda = \Delta_{1-\lambda}$
- $\widehat{\Delta}1 = 0$.

This operator has an appearance (1.4).

1.4 Equivariant maps between modules of differential operators.

The pencil of second order operators considered above defines the maps $T_{\lambda,\mu}$ between module of second order operators on densitites of weight λ to module of second order operators of weight μ.

The Lie derivative of self-adjoint operator is self-adjoint. Hence uniqueness implies that:

$$\mathrm{ad}_{\mathbf{K}} \circ T_{\lambda,\mu} = T_{\lambda,\mu}\mathrm{ad}_{\mathbf{K}}$$

This implies very important

Corollary. $T_{\lambda,\mu}$ is equivariant map

$$\Delta_\lambda \xrightarrow{T_{\lambda\mu}} \Delta_\lambda \tag{1.5}$$

for $\lambda, \mu \neq 0, 1/2, 1$. The "bare hand" proof is difficult....

This map has the following appearance:

If an operator $\Delta_\lambda \in \mathcal{D}_\lambda(M)$ is given in local coordinates by the expression $\Delta_\lambda = A^{ij}(x)\partial_i\partial_j + A^i(x)\partial_i + A(x)$ then its image $T_{\lambda,\mu}(\Delta_\lambda) = \Delta_\mu \in \mathcal{D}_\mu(M)$ is given in the same local coordinates by the expression $\Delta_\mu = B^{ij}(x)\partial_i\partial_j + B^i(x)\partial_i + B(x)$ where

$$\begin{cases} B^{ij} & = A^{ij}\,, \\ B^i & = \frac{2\mu-1}{2\lambda-1}A^i + \frac{2(\lambda-\mu)}{2\lambda-1}\partial_j A^{ji}\,, \\ B & = \frac{\mu(\mu-1)}{\lambda(\lambda-1)}A + \frac{\mu(\lambda-\mu)}{(2\lambda-1)(\lambda-1)}\left(\partial_j A^j - \partial_i\partial_j A^{ij}\right)\,. \end{cases} \tag{1.6}$$

At the exceptional cases $\lambda, \mu = 0, \frac{1}{2}, 1$, non-isomorphic modules occur.

Remak It was calculated by Duval and Ovsienko (see [3] and [8] in a straightforward way. Equations (1.4) and (1.5) illuminate this result.)

Very beautiful example (Mathonet, Lecomte:)

$$T_{\lambda,\mu}\left(\mathcal{L}^{(\lambda)}_{\mathbf{X}} \circ \mathcal{L}^{(\lambda)}_{\mathbf{Y}}\right) = \mathcal{L}^{(\mu)}_{\mathbf{X}} \circ \mathcal{L}^{(\mu)}_{\mathbf{Y}} + \frac{\mu-\lambda}{2\lambda-1}\mathcal{L}_{[\mathbf{X},\mathbf{Y}]}$$

Our beautiful proof:

$$\mathcal{L}^{(\lambda)}_{\mathbf{X}} \circ \mathcal{L}^{(\lambda)}_{\mathbf{Y}} = \frac{1}{2}\left[\mathcal{L}^{(\lambda)}_{\mathbf{X}} \circ \mathcal{L}^{(\lambda)}_{\mathbf{Y}} + \mathcal{L}^{(\lambda)}_{\mathbf{X}} \circ \mathcal{L}^{(\lambda)}_{\mathbf{Y}}\right]\frac{1}{2}\left[\mathcal{L}^{(\lambda)}_{\mathbf{X}} \circ \mathcal{L}^{(\lambda)}_{\mathbf{Y}} - \mathcal{L}^{(\lambda)}_{\mathbf{X}} \circ \mathcal{L}^{(\lambda)}_{\mathbf{Y}}\right] +$$

The first operator is self-adjoint, the second antiself-adjoint hence we draw the following self-adjoint pencil through this operator

$$\widehat{\Delta} = \frac{1}{2}\left[\mathcal{L}_{\mathbf{X}} \circ \mathcal{L}_{\mathbf{Y}} + \mathcal{L}_{\mathbf{X}} \circ \mathcal{L}_{\mathbf{Y}}\right] + \frac{1}{2}\frac{2\widehat{\lambda}-1}{2\lambda-1}\left[\mathcal{L}_{\mathbf{X}} \circ \mathcal{L}_{\mathbf{Y}} - \mathcal{L}_{\mathbf{X}} \circ \mathcal{L}_{\mathbf{Y}}\right] =$$

$$\frac{1}{2}\left[\mathcal{L}_{\mathbf{X}} \circ \mathcal{L}_{\mathbf{Y}} + \mathcal{L}_{\mathbf{X}} \circ \mathcal{L}_{\mathbf{Y}}\right] + \frac{2\widehat{\lambda}-1}{4\lambda-2}\mathcal{L}_{[\mathbf{X},\mathbf{Y}]}$$

We see that $\widehat{\Delta}\big|_{\widehat{\lambda}=\lambda} = \mathcal{L}^{(\lambda)}_{\mathbf{X}} \circ \mathcal{L}^{(\lambda)}_{\mathbf{Y}}$ and

$$\widehat{\Delta}\big|_{\widehat{\lambda}=\mu} = \frac{1}{2}\left[\mathcal{L}^{(\mu)}_{\mathbf{X}} \circ \mathcal{L}^{(\mu)}_{\mathbf{Y}} + \mathcal{L}^{(\mu)}_{\mathbf{X}} \circ \mathcal{L}^{(\mu)}_{\mathbf{Y}}\right] + \frac{1}{2}\left[\mathcal{L}^{(\mu)}_{\mathbf{X}} \circ \mathcal{L}^{(\mu)}_{\mathbf{Y}} - \mathcal{L}^{(\mu)}_{\mathbf{X}} \circ \mathcal{L}^{(\mu)}_{\mathbf{Y}}\right] - \frac{1}{2}\mathcal{L}^{(\mu)}_{[\mathbf{X},\mathbf{Y}]} + \frac{2\mu-1}{4\lambda-2}\mathcal{L}^{(\mu)}_{[\mathbf{X},\mathbf{Y}]}$$

$$= \mathcal{L}^{(\mu)}_{\mathbf{X}} \circ \mathcal{L}^{(\mu)}_{\mathbf{Y}} + \frac{\mu - \lambda}{2\lambda - 1} \mathcal{L}^{(\mu)}_{[\mathbf{X},\mathbf{Y}]} \blacksquare$$

Proof of the theorem. First we construct Beltrami-Laplas pencil (1.3) such that second order terms coincide, then add the operator $(2\widehat{\lambda}-1)\mathcal{L}_{\mathbf{X}}$ choosing $\mathbf{X}$ such that first terms coicide, then add scalar. Thus we come to self-adjoint pencil $\widehat{\Delta}$ passing through the operator.

Most important to prove the uniqueness (Without uniqueness we will not have the equivariant map (1.5)).

If $\widehat{\Delta}'$ is an arbitrary operator which obeys the condition of theorem then

$$\widehat{\Delta}' = \widehat{\Delta} + (\widehat{\lambda} - \lambda_0)(L_1 + \widehat{\lambda} F)$$

Step by step we prove that it vanishes.

1.5 Canonical pencil in general case

:

$$\Delta = t^\delta \left[S^{ik}\partial_i\partial_k + \left(\partial_r S^{ri} + (2\lambda + \delta - 1)\gamma^i\right)\partial_i + \left(\lambda\partial_i\gamma^i + \lambda(\lambda = \delta - 1)\theta\right)\right]$$

- $\mathbf{S} = t^\delta S^{ab}(x) = S^{ab}(x)\mathcal{D}x^\delta$ is symmetric contravariant tensor field-density of the weight δ. Under changing of local coordinates $x^{a'} = x^{a'}(x^a)$ it transforms in the following way:

$$S^{a'b'} = J^{-\delta} x^{a'}_a x^{b'}_b S^{ab} \,,$$

- γ^a is a symbol of upper connection-density of weight δ. Under changing of local coordinates $x^{a'} = x^{a'}(x^a)$ it transforms in the following way:

$$\gamma^{a'} = J^{-\delta} x^{a'}_a \left(\gamma^a + S^{ab}\partial_b \log J\right) \,,$$

- and θ transforms in the following way:

$$\theta' = J^{-\delta}\left(\theta + 2\gamma^a\partial_a \log J + \partial_a \log J\, S^{ab}\partial_b \log J\right)$$

  ($\theta = \gamma^a\gamma_a +$ scalar in the case if symbol is invertible)

1.6 Special cases.

Consider $\lambda = \frac{1-\delta}{2}$. Then

$$\Delta = t^\delta \left[S^{ik}\partial_i\partial_k + \partial_r S^{ri}\partial_i + \frac{1-\delta}{2}\left(\partial_i\gamma^i + \frac{\delta-1}{2}\theta\right)\right]$$

Example. $\delta = 0$. S^{ik} defines Poisson structure. —Batalin-Vilkovisky formalism.

Next example $\delta = 2$, $n = 1$. $S = 1$ (invariant) We come to

$$\Delta = t^2\left[\partial_x^2 - \frac{1}{2}U(x)\right] = |Dx|^2\left[\partial_x^2 - \frac{1}{2}U(x)\right],$$

$$\Delta\left(\Psi(x)|Dx|^{-\frac{1}{2}}\right) = \left(\Psi_{xx}(x) - \frac{1}{2}U(x)\Psi(x)\right)|Dx|^{\frac{3}{2}}. \qquad (1.7)$$

This operator leads us to .... Schwarzian.

2 Schwarzian , Projective geometry

See how operator (1.7)transforms the operator above under diffeomorphisms.

If $f = y(x)$ is diffeomorphism then

$$\Delta^f\left(\Psi(x)|Dx|^2\right) = \left\{|Dy|^2\left[\partial_y^2 - \frac{1}{2}U(y)\right]\left[\Psi\left(x(y)\right)|Dx|^{-\frac{1}{2}}\right]\right\}\big|_{y=y(x)} =$$

$$\left\{|Dy|^2\left[\partial_y^2 - \frac{1}{2}U(y)\right]\left[\Psi\left(x(y)\right)x_y^{-\frac{1}{2}}|Dy|^{-\frac{1}{2}}\right]\right\}\big|_{y=y(x)} =$$

$$\left[\Psi_{xx}(x)x_y^{\frac{3}{2}} + \frac{3}{4}\Psi(x)x_y^{-\frac{5}{2}}x_{yy}^2 - \frac{1}{2}\Psi(x)x_y^{-\frac{3}{2}}x_{yyy} - \frac{1}{2}\left[U(y(x)) +\right]\Psi(x)\right]y_x^{\frac{3}{2}}|Dx|^{\frac{3}{2}} =$$

$$\left[\Psi_{xx}(x) + \frac{3}{4}\Psi(x)x_y^{-4}x_{yy}^2 - \frac{1}{2}\Psi(x)x_y^{-3}x_{yyy} - \frac{1}{2}U(y(x))\Psi(x)\right]|Dx|^{\frac{3}{2}} =$$

$$\left[\Psi_{xx}(x) + \frac{3}{4}\Psi(x)x_y^{-4}x_{yy}^2 - \frac{1}{2}\Psi(x)x_y^{-3}x_{yyy} - \frac{1}{2}U(y(x))\Psi(x)\right]|Dx|^{\frac{3}{2}} =$$

$$\left[\Psi_{xx}(x) + \frac{3}{4}\Psi(x)x_y^{-4}x_{yy}^2 - \frac{1}{2}\Psi(x)x_y^{-3}x_{yyy} - \frac{1}{2}\left[U(y(x)) + \left(\frac{x_{yyy}}{x_y} - \frac{3}{2}\frac{x_{yy}^2}{x_y^2}\right)y_x^2\right]\Psi(x)\right]|Dx|^{\frac{3}{2}} =$$

Sorry for these calculations but *Paris vaut bien une messe*:

$$\Delta^f = \Delta - \frac{1}{2}\left(\frac{x_{yyy}}{x_y} - \frac{3}{2}\frac{x_{yy}^2}{x_y^2}\right)|Dy|^2 \tag{2.1}$$

The cocycle

$$\mathcal{S}(f^{-1}) = \left(\frac{x_{yyy}}{x_y} - \frac{3}{2}\frac{x_{yy}^2}{x_y^2}\right)|Dy|^2 \tag{2.2}$$

is Schwarzian.

2.1 Schwartzian and second order connection

The "potential" U in the second order Sturm-Lioville operator of weight $\delta = 2$ (1.7) on $\mathbf{R}$ plays the role of second order connection. Its transformation is defined by Schwarzian. (Compare with first order operator defined by usual connection.)

2.2 Schwarzian and ... normal gauge

Valya Ovsienko likes to say that Schwarzian has more than 600 different manifestations. Another day I discussed with Adam Biggs one of them. Here is the result of our discussions.

Schwarzian and ... normal gauging conditions

Normal gauging is tremendously powerfull tool in geometry. E.g. the quickest way to define invariants of gauge field (connection) is to consider connection $A_\mu(x)$ in so called "normal gauge":

$$A'_\mu(x)\colon \qquad A_\mu(x)(x^\mu - x^\mu_0) = 0\,, \quad (A'_\mu = gA_\mu g^{-1} + g^{-1}\partial_\mu g)\,. \tag{1}$$

(Here A_μ takes values in the Lie algebra Lie group $\mathcal{G}$, $g(x)$ is the function in G.) Coefficients of Taylor series expansion are curvature of connection and its covariant derivatives at the point $\mathbf{p}$. (Here x^μ are local coordiantes in the vicinity of the point $\mathbf{p}$ with coordinates x^μ_0.) E.g. if $A_\mu(x)$ is electromagnetic field given in a vicinity of the point $x^\mu_0 = 0$ in normal gauge then $A_\mu(x) = F_{\mu\nu}x^\nu + \dots$ where $F_{\mu\nu}$ is the value of electromagnetic field tensor. Another example: if Riemannian metric is given in normal coordinates: $g_{\mu\nu}x^\nu = \delta_{\mu_\nu}x^\nu$ then

$$g_{\mu\nu}(x) = \delta_{\mu\nu} + \dots R_{\mu\alpha\nu\beta}(x^\alpha - x^\alpha_0)(x^\beta - x^\beta_0) + \dots$$

where $R_{\mu\alpha\nu\beta}$ is curvature tensor at the point $\mathbf{p}$. The normal gauging condition $g_{\mu\nu}x^\nu = \delta_{\mu_\nu}x^\nu$ is strictly related with condition $\Gamma^\alpha_{\mu\nu}x^\mu x^\nu = 0$ for geodesic coordinates. You can read about normal gauge in different textbooks [1].

Now *revenons a nos moutons*. Let x be a local coordinate on projective line $P\mathbf{R}^1$ which fixes projective structure in a vicinity of the point $\mathbf{p}$: one admits the changing of coordinates $x \mapsto \frac{ax+b}{cx+d}$. Let $F(x)$ be a local expression for diffeomorphism of $P\mathbf{R}^1$.

Recall that Shwarzian equals to the following density of the weight 2:

$$\mathcal{S}^F = \left(\frac{F_{xxx}}{F_x} - \frac{3}{2}\frac{F_{xx}^2}{F_x^2}\right)|Dx|^2 . \tag{2}$$

This is non-trivial 1-cocycle of diffeomorphisms which vanishes on projective transformations. Projective transformations have three degrees of freedom. Consider "normal" gauging of the diffeomorphism: the new diffeomorphism F' such that it difffers form F on projective transformation and F' is identity in a vicinity of $\mathbf{p}$ up to the third order terms: $F'\colon F = G \circ F'$ where $G = \frac{ax+b}{cx+d}$ is projective transformation such that

$$F'(x) = (x-x_{_0}) + O((x-x_{_0})^3), : \qquad F'\big|_{x=\mathbf{p}} = \frac{dF'(x)}{dx}\big|_{x=\mathbf{p}} = \frac{d^2F'(x)}{dx^2}\big|_{x=\mathbf{p}} = 0 .$$

The value of gauged diffeomorphsim F' at the point $\mathbf{p}$ is the Schwarzian of F at the point $\mathbf{p}$. Calculate it.

If $F(x) = a + b(x-x_{_0}) + c(x-x_{_0})^2 + d(x-x_{_0})^3 + o((x-x_{_0})^3)$ then consider composition of projective tansformations such that they "kill" derivatives: $G_1\colon x \mapsto x - a$ (translation), $G_2\colon x \mapsto x \mapsto \frac{x}{b}$ and special projective transformation $G_3\colon x \mapsto \frac{x}{1+px}$ with $p = \frac{c}{b}$. Then we come to

$$F'(x) = G_3 \circ G_2 \circ G_1 \circ F = \frac{(x-x_{_0}) + \frac{c}{b}(x-x_{_0})^2 + \frac{d}{b}(x-x_{_0})^3 + o((x-x_0)^3)}{1 + p((x-x_{_0}) + \frac{c}{b}(x-x_{_0})^2 + o((x-x_{_0})^2))} =$$

$$(x-x_0)^2 + d'(x-x_0)^3 + o((x-x_0)^3,$$

where

$$d' = \frac{d}{b} - \frac{c^2}{b^2} = \left(\frac{6F_{xxx}}{F_x} - \frac{(2F_{xx})^2}{F_x^2}\right)\big|_{x=x_0} = 6\left(\frac{F_{xxx}}{F_x} - \frac{3}{2}\frac{F_{xx}^2}{F_x^2}\right)\big|_{x=x_0} .$$

We come (up to a multiplier) to the Schwarzian (2). Schwarzian appears in the way as curvature of gauge field....

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[1] One can find the excellent exposition of the geometry of normal gauging in one of Appendices of the famous article: "On the heat equation and the index theorem" of M. Atiyah, R. Bott and V.K. Patodi)

2.3 Projective structures on curves in $\mathbf{R}P^n$ and Schwarzian. Curves in $\mathbf{R}P^n$ and $n+1$-th order operators.

We consider the following construction.

let $C\colon \quad u^i(x)$ be an arbitrary curve in $\mathbf{R}^{n+1}$. Let $\boldsymbol{\rho}$ be an arbitrary density and ∇ an arbitrary connection.

Let $\mathcal{D} = |Dx|(\partial_x + \gamma)$

We consider the following pencil of operators For an arbitrary density $\boldsymbol{s} = s(x)|Dx|^\lambda$ of weight λ we consider operator

$$\Delta_\lambda(\boldsymbol{s}) = \frac{\det \begin{pmatrix} u^1\boldsymbol{\rho}^\lambda & u^2\boldsymbol{\rho}^\lambda & \dots u^{n+1}\boldsymbol{\rho}^\lambda & \boldsymbol{s}_\lambda \\ \mathcal{D}u^1\boldsymbol{\rho}^\lambda & \mathcal{D}u^2\boldsymbol{\rho}^\lambda & \dots \mathcal{D}u^{n+1}\boldsymbol{\rho}^\lambda & \mathcal{D}\boldsymbol{s}_\lambda \\ \mathcal{D}^2u^1\boldsymbol{\rho}^\lambda & \mathcal{D}^2u^2\boldsymbol{\rho}^\lambda & \dots \mathcal{D}^2u^{n+1}\boldsymbol{\rho}^\lambda & \mathcal{D}^2\boldsymbol{s}_\lambda \\ \dots & \dots & \dots\dots & \dots \\ \mathcal{D}^{n+1}u^1\boldsymbol{\rho}^\lambda & \mathcal{D}^{n+1}u^2\boldsymbol{\rho}^\lambda & \dots \mathcal{D}^2u^{n+1}\boldsymbol{\rho}^\lambda & \mathcal{D}^{n+1}\boldsymbol{s}_\lambda \end{pmatrix}}{\det \begin{pmatrix} u^1\boldsymbol{\rho}^\lambda & u^2\boldsymbol{\rho}^\lambda & \dots u^{n+1}\boldsymbol{\rho}^\lambda \\ \mathcal{D}u^1\boldsymbol{\rho}^\lambda & \mathcal{D}u^2\boldsymbol{\rho}^\lambda & \dots \mathcal{D}u^{n+1}\boldsymbol{\rho}^\lambda \\ \mathcal{D}^2u^1\boldsymbol{\rho}^\lambda & \mathcal{D}^2u^2\boldsymbol{\rho}^\lambda & \dots \mathcal{D}^2u^{n+1}\boldsymbol{\rho}^\lambda \\ \dots & \dots & \dots\dots \\ \mathcal{D}^n u^1\boldsymbol{\rho}^\lambda & \mathcal{D}^n u^2\boldsymbol{\rho}^\lambda & \dots \mathcal{D}^2u^n\boldsymbol{\rho}^\lambda q \end{pmatrix}} \tag{2.3}$$

This operator sends density of weight λ to the density of the weight $\lambda+n+1$

E.g. for curve $C\colon u(x), v(x)$ in $\mathbf{R}^2$

$$\Delta_\lambda(\boldsymbol{s}) = \frac{\det \begin{pmatrix} u\boldsymbol{\rho}^\lambda & v\boldsymbol{\rho}^\lambda & \boldsymbol{s}_\lambda \\ \mathcal{D}u\boldsymbol{\rho}^\lambda & \mathcal{D}v\boldsymbol{\rho}^\lambda & \mathcal{D}\boldsymbol{s}_\lambda \\ \mathcal{D}^2u\boldsymbol{\rho}^\lambda & \mathcal{D}^2v\boldsymbol{\rho}^\lambda & \mathcal{D}^2\boldsymbol{s}_\lambda \end{pmatrix}}{\det \begin{pmatrix} u\boldsymbol{\rho}^\lambda & v\boldsymbol{\rho}^\lambda \\ \mathcal{D}u\boldsymbol{\rho}^\lambda & \mathcal{D}v\boldsymbol{\rho}^\lambda \end{pmatrix}} \tag{2.4}$$

This operator sends density of weight λ to the density of the weight $\lambda+2$

Densities proportional to linear combination of functions $u^i(t)$ belong to kernel.

Remark Note that in components all terms proportional to γ and $\boldsymbol{\rho}$ disappear....

Consider the projection of curve C in $\mathbf{R}P^n$.

Denominator is the density of the weight $n\lambda+1+\dots+n = (n+1)\lambda+\frac{n(n+1}{2)}$. We may choose multiplier such that for $\lambda = -\frac{1}{n+1}$ denominator equals to 1.

We come to Schwarzian in third terms for operator...

2.4 (Anti)-self adjoint operator of order n on $\mathbf{R}$

.

Example

$$\Delta = t^n \mathcal{D}^n, \qquad \text{where } \mathcal{D} = t^n(\partial_x + \widehat{\lambda}\gamma_{\text{-}}) \tag{2.5}$$

Calculations show that...

Example

$$\Delta = t^{n+\delta'} \left[s(x)\partial_x^n + \frac{n}{2}(s_x + 2s\gamma_{\text{-}}\widehat{\lambda}_{n+\delta'})\partial^{n-1} + B_n\partial_x^{n-1} + \dots \right]$$

where

$$B_n = \frac{n(n-1)}{2}\left[\gamma_x^{\text{-}} + s(\gamma_{\text{-}}^2 + \tau)\right] + \frac{n(n-1)}{2}\left[\frac{n-2}{6}s_x\gamma_{\text{-}} - \frac{n+1+3\delta'}{6}sU_\gamma - \frac{\delta'^2}{4}s\gamma^2 + s\rho\right] \tag{2.6}$$

Here ρ, τ are densities of weight 2, $U_\gamma = \gamma_x - \frac{1}{2}\gamma^2$ is related with Schwarzian "antiderivative" $\widehat{\lambda}_k = \widehat{\lambda} + \frac{k-1}{2}$

We see here how Schwarzian appears in critical dimensions....

2.5 Transvectants and symplectic geometry

We will expose here the construction of ovsienko and collaborators (see e.g.[8] using little bit different point of view.

Let (x, t) be coordinates on $\widehat{\mathbf{R}}$ and let (x, ξ) be coordinates on the cotangent bundle $T^*\mathbf{R}$ ($\xi \approx \partial_x$). We see that $\xi \approx \frac{1}{t}$.

Under suitable identification of $\widehat{\mathbf{R}}$ and cotangent bundle $T^*\mathbf{R}$ linear canonical transformations = projective transformations. Namely:

Projective transformations algebra acting on $\mathbf{R}$ is spanned by

$$\left\{\frac{\partial}{\partial x}, x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} = x\frac{\partial}{\partial x} + \widehat{\lambda}, x^2\frac{\partial}{\partial x} + 2xt\frac{\partial}{\partial t} = x^2\frac{\partial}{\partial x} + 2x\widehat{\lambda}\right\}$$

$$\downarrow$$

Projective transformations algebra acting on $\widehat{\mathbf{R}}$ is spanned by the vectors

$$\left\{\frac{\partial}{\partial x}, x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} = x\frac{\partial}{\partial x} + \widehat{\lambda}, x^2\frac{\partial}{\partial x} + 2xt\frac{\partial}{\partial t} = x^2\frac{\partial}{\partial x} + 2x\widehat{\lambda}\right\}$$

$$\updownarrow$$

Corresponding symplectic transformations algebra acting on $\widehat{\mathbf{R}}$ is spanned by the vectors

$$\left\{\frac{\partial}{\partial x}, x\frac{\partial}{\partial x} - \xi\frac{\partial}{\partial \xi} = x\frac{\partial}{\partial x} + \widehat{\lambda}, x^2\frac{\partial}{\partial x} - 2x\xi\frac{\partial}{\partial \xi} = x^2\frac{\partial}{\partial x} + 2x\widehat{\lambda}\right\}$$

Hamiltonians of these vetor fiels are

$$\{\xi, \xi x, \xi x^2\}$$

$$dxd\xi = dpd\xi$$

$$\begin{cases} \xi = \frac{p^2}{2} \\ x = \frac{q}{p} \end{cases} \qquad \begin{cases} t = \frac{2}{p^2} \\ x = \frac{q}{p} \end{cases}$$

volume form associated with scalar product on $\mathcal{F}(\mathbf{R})$ is $\dfrac{dxdt}{t^2} = dxd\xi$ symplectic 2-form

Finally for transvectants are expressed in terms of iterated Poisson bracket

$$B_m(F,G) = \operatorname{Tr} P^m(F \otimes G)$$

where $P(F \otimes G) = F_\xi \otimes G_x - F_x \otimes G_\xi$:

$B_1(F,G) = F_\xi G_x - F_x G_\xi \{F,G\}$, $B_2(F,G)) = F_{\xi\xi}G_{xx} - 2F_{x\xi}G_{x\xi} + F_{xx}G_{\xi\xi}, ...$

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