Differential operators on algebra of densities; from Schwarzian derivative to Batalin-Vilkovisky operator

Hovhannes Khudaverdian

University of Manchester, Manchester, UK

Leibniz Universitat Hannover & Yerevan State University
Supersymmetry in Integrable Systems - SIS'11
International Workshop, 01-04 August 2011
Hannover, Germany

Contents

Schwarzian derivative. First encounter

For function $f = f(x)$ Schwarzian derivative

$$f(x) \mapsto (\mathcal{S} f)(x) = \left(\frac{f'''(x)}{f'(x)}\right) - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2$$

It is the famous construction in projective geometry

$$(\mathcal{S} f)(x) \equiv 0 \Leftrightarrow f \text{ is projective transformation, } \left(f = \frac{ax+b}{cx+d}\right)$$

Schwarzian derivative. First encounter

For function $f = f(x)$ Schwarzian derivative

$$f(x) \mapsto (\mathcal{S}f)(x) = \left(\frac{f'''(x)}{f'(x)}\right) - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2$$

It is the famous construction in projective geometry

$$(\mathcal{S}f)(x) \equiv 0 \Leftrightarrow f \text{ is projective transformation, } \left(f = \frac{ax+b}{cx+d}\right)$$

$$\mathcal{S}(f \circ g) = \mathcal{S}(f) \circ g + \mathcal{S}(g)$$

Schwarzian derivative. First encounter

For function $f = f(x)$ Schwarzian derivative

$$f(x) \mapsto (\mathscr{S}f)(x) = \left(\frac{f'''(x)}{f'(x)}\right) - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2$$

It is the famous construction in projective geometry

$$(\mathscr{S}f)(x) \equiv 0 \Leftrightarrow f \text{ is projective transformation, } \left(f = \frac{ax+b}{cx+d}\right)$$

$$\mathscr{S}(f \circ g) = \mathscr{S}(f) \circ g + \mathscr{S}(g) \text{ cocycle condition}$$

Recall densities

A density of the weight λ on (super)manifold M—$s(x)|Dx|^\lambda$.
Under a change of coordinates it is multiplied by the λ-th power of the Jacobian of the coordinate transformation:

$$s(x)|Dx|^\lambda = s(x(x')) \left|\frac{Dx}{Dx'}\right|^\lambda |Dx'|^\lambda = s(x(x')) \left(\det\left(\frac{\partial x}{\partial x'}\right)\right)^\lambda |Dx'|^\lambda .$$

Recall densities

A density of the weight λ on (super)manifold M—$s(x)|Dx|^{\lambda}$.

Under a change of coordinates it is multiplied by the λ-th power of the Jacobian of the coordinate transformation:

$$s(x)|Dx|^{\lambda} = s(x(x')) \left|\frac{Dx}{Dx'}\right|^{\lambda} |Dx'|^{\lambda} = s(x(x')) \left(\det\left(\frac{\partial x}{\partial x'}\right)\right)^{\lambda} |Dx'|^{\lambda}.$$

$$\left|\frac{Dx}{Dx'}\right| = \det \frac{\partial x}{\partial x'}$$

Recall densities

A density of the weight λ on (super)manifold M—$s(x)|Dx|^\lambda$.
Under a change of coordinates it is multiplied by the λ-th power of the Jacobian of the coordinate transformation:

$$s(x)|Dx|^\lambda = s(x(x')) \left|\frac{Dx}{Dx'}\right|^\lambda |Dx'|^\lambda = s(x(x')) \left(\det\left(\frac{\partial x}{\partial x'}\right)\right)^\lambda |Dx'|^\lambda .$$

$$\left|\frac{Dx}{Dx'}\right| = \det\frac{\partial x}{\partial x'}$$

it is for usual manifolds

Recall densities

A density of the weight λ on (super)manifold M—$s(x)|Dx|^\lambda$.
Under a change of coordinates it is multiplied by the λ-th power of the Jacobian of the coordinate transformation:

$$s(x)|Dx|^\lambda = s(x(x'))\left|\frac{Dx}{Dx'}\right|^\lambda |Dx'|^\lambda = s(x(x'))\left(\det\left(\frac{\partial x}{\partial x'}\right)\right)^\lambda |Dx'|^\lambda .$$

$$\left|\frac{Dx}{Dx'}\right| = \det\frac{\partial x}{\partial x'}$$ it is for usual manifolds

For supermanifolds $x \mapsto (x,\theta)$, $x' \mapsto (x',\theta')$

$$\left|\frac{D(x,\theta)}{D(x',\theta')}\right| = \mathrm{Ber}\,\frac{\partial(x,\theta)}{\partial(x',\theta')} = \frac{\det\left(\frac{\partial x}{\partial x'} - \frac{\partial \theta}{\partial x'}\left(\frac{\partial \theta}{\partial \theta'}\right)^{-1}\frac{\partial x}{\partial \theta'}\right)}{\det\left(\frac{\partial \theta}{\partial \theta'}\right)}$$

Examples of densities

Density of weight $\lambda = 0$ is a usual scalar function.
Density of weight $\lambda = 1$ is a volume form.
Wave function Ψ is a density of weight $\lambda = \frac{1}{2}$ (semidensity):

$$\Psi(x)\sqrt{Dx} = \Psi(x(x'))\sqrt{\det\left(\frac{\partial x}{\partial x'}\right)}\sqrt{Dx'}$$

Examples of densities

Density of weight $\lambda = 0$ is a usual scalar function.
Density of weight $\lambda = 1$ is a volume form.
Wave function Ψ is a density of weight $\lambda = \frac{1}{2}$ (semidensity):

$$\Psi(x)\sqrt{Dx} = \Psi(x(x'))\sqrt{\det\left(\frac{\partial x}{\partial x'}\right)}\sqrt{Dx'}$$

or in supercase:

$$\Psi(x,\theta)\sqrt{D(x,\theta)} = \Psi(x(x',\theta'),\theta(x',\theta'))\sqrt{\mathrm{Ber}\,\frac{\partial(x,\theta)}{\partial(x',\theta')}}\sqrt{D(x',\theta')}$$

Examples of densities

Density of weight $\lambda = 0$ is a usual scalar function.
Density of weight $\lambda = 1$ is a volume form.
Wave function Ψ is a density of weight $\lambda = \frac{1}{2}$ (semidensity):

$$\Psi(x)\sqrt{Dx} = \Psi(x(x'))\sqrt{\det\left(\frac{\partial x}{\partial x'}\right)}\sqrt{Dx'}$$

or in supercase:

$$\Psi(x,\theta)\sqrt{D(x,\theta)} = \Psi(x(x',\theta'),\theta(x',\theta'))\sqrt{\mathrm{Ber}\,\frac{\partial(x,\theta)}{\partial(x',\theta')}}\sqrt{D(x',\theta')}$$

Remark We suppose that (super)manifold is orientable and the orientation is chosen; Jacobians of all coordinate transformations are positive.

Sturm-Liouville operator on densities of chosen weights

$$\mathscr{F}_\lambda(M) = \{\text{space of densities of the weight } \lambda \text{ on manifold } M\}$$

Sturm-Liouville operator on densities of chosen weights

$$\mathcal{F}_\lambda(M) = \{\text{space of densities of the weight } \lambda \text{ on manifold } M\}$$

Let $M = \mathbf{R}$. Consider the second order operator $\Delta\colon \mathcal{F}_{-\frac{1}{2}}(\mathbf{R}) \to \mathcal{F}_{+\frac{3}{2}}(\mathbf{R})$ such that

$$\Delta\left(\Psi(x)|Dx|^{-\frac{1}{2}}\right) = \left(\frac{\partial^2\Psi(x)}{\partial x^2} + U(x)\Psi(x)\right)|Dx|^{+\frac{3}{2}}$$

Δ is Sturm-Lioville operator of the weight 2:

$$\Delta = |Dx|^2\left(\frac{\partial^2}{\partial x^2} + U(x)\right).$$

Sturm-Liouville operator on densities of chosen weights

$$\mathscr{F}_\lambda(M) = \{\text{space of densities of the weight } \lambda \text{ on manifold } M\}$$

Let $M = \mathbf{R}$. Consider the second order operator $\Delta\colon \mathscr{F}_{-\frac{1}{2}}(\mathbf{R}) \to \mathscr{F}_{+\frac{3}{2}}(\mathbf{R})$ such that

$$\Delta\left(\Psi(x)|Dx|^{-\frac{1}{2}}\right) = \left(\frac{\partial^2\Psi(x)}{\partial x^2} + U(x)\Psi(x)\right)|Dx|^{+\frac{3}{2}}$$

Δ is Sturm-Lioville operator of the weight 2:

$$\Delta = |Dx|^2\left(\frac{\partial^2}{\partial x^2} + U(x)\right).$$

Why this operator?

Sturm-Liouville operator on densities of chosen weights

$$\mathcal{F}_\lambda(M) = \{\text{space of densities of the weight } \lambda \text{ on manifold } M\}$$

Let $M = \mathbf{R}$. Consider the second order operator
$\Delta\colon \mathcal{F}_{-\frac{1}{2}}(\mathbf{R}) \to \mathcal{F}_{+\frac{3}{2}}(\mathbf{R})$ such that

$$\Delta\left(\Psi(x)|Dx|^{-\frac{1}{2}}\right) = \left(\frac{\partial^2\Psi(x)}{\partial x^2} + U(x)\Psi(x)\right)|Dx|^{+\frac{3}{2}}$$

Δ is Sturm-Lioville operator of the weight 2:

$$\Delta = |Dx|^2\left(\frac{\partial^2}{\partial x^2} + U(x)\right).$$

Why this operator? Why so strange choice of weights?

Transformation of the operator Δ under diffeomorphism of **R**

Consider $f: y = y(x)$ (with $y_x > 0$). We come to $\Delta^{(f)} = f^* \circ \Delta \circ \left(f^{-1}\right)^*$ such that

$$\Delta^{(f)}\left(\Psi(x)|Dx|^{-\frac{1}{2}}\right) = \left[\left(\frac{\partial^2}{\partial y^2} + U(y)\right)\left(\Psi(x(y))\left|\frac{\partial x}{\partial y}\right|^{-\frac{1}{2}}\right)|Dy|^{\frac{3}{2}}\right]_{y=f(x)}$$

Transformation of the operator Δ under diffeomorphism of $\mathbf{R}$

Consider $f: y = y(x)$ (with $y_x > 0$). We come to $\Delta^{(f)} = f^* \circ \Delta \circ \left(f^{-1}\right)^*$ such that

$$\Delta^{(f)}\left(\Psi(x)|Dx|^{-\frac{1}{2}}\right) = \left[\left(\frac{\partial^2}{\partial y^2} + U(y)\right)\left(\Psi(x(y))\left|\frac{\partial x}{\partial y}\right|^{-\frac{1}{2}}\right)|Dy|^{\frac{3}{2}}\right]_{y=f(x)}$$

$$\left(\left(\Psi x_y^{-\frac{1}{2}}\right)_{yy} + U\Psi x_y^{-\frac{1}{2}}\right) y_x^{3/2}|Dx|^{\frac{3}{2}} =$$

Transformation of the operator Δ under diffeomorphism of $\mathbf{R}$

Consider $f: y = y(x)$ (with $y_x > 0$). We come to $\Delta^{(f)} = f^* \circ \Delta \circ \left(f^{-1}\right)^*$ such that

$$\Delta^{(f)}\left(\Psi(x)|Dx|^{-\frac{1}{2}}\right) = \left[\left(\frac{\partial^2}{\partial y^2} + U(y)\right)\left(\Psi(x(y))\left|\frac{\partial x}{\partial y}\right|^{-\frac{1}{2}}\right)|Dy|^{\frac{3}{2}}\right]_{y=f(x)}$$

$$\left(\left(\Psi x_y^{-\frac{1}{2}}\right)_{yy} + U\Psi x_y^{-\frac{1}{2}}\right) y_x^{3/2}|Dx|^{\frac{3}{2}} =$$

$$= \left[\Psi_{xx} + \left(U\right.\right.$$

Transformation of the operator Δ under diffeomorphism of $\mathbf{R}$

Consider $f: y = y(x)$ (with $y_x > 0$). We come to $\Delta^{(f)} = f^* \circ \Delta \circ \left(f^{-1}\right)^*$ such that

$$\Delta^{(f)}\left(\Psi(x)|Dx|^{-\frac{1}{2}}\right) = \left[\left(\frac{\partial^2}{\partial y^2} + U(y)\right)\left(\Psi(x(y))\left|\frac{\partial x}{\partial y}\right|^{-\frac{1}{2}}\right)|Dy|^{\frac{3}{2}}\right]_{y=f(x)}$$

$$\left(\left(\Psi x_y^{-\frac{1}{2}}\right)_{yy} + U\Psi x_y^{-\frac{1}{2}}\right) y_x^{3/2} |Dx|^{\frac{3}{2}} =$$

$$= \left[\Psi_{xx} + \left(U - \frac{1}{2}\underbrace{\left(\frac{x_{yyy}}{x_y} - \frac{3x_{yy}^2}{2x_y^2}\right)}_{\text{Schwarzian derivative of } y(x)}\right) y_x^2\right] |Dx|^{\frac{3}{2}}$$

Comparison of operators Δ and Δ^f.

We see that for a diffeomorphism $f : y = f(x)$

$$\Delta^{(f)} = |Dx|^2 \frac{\partial^2}{\partial x^2} + |Dx|^2 U^{(f)}(x),$$

where $|Dx|^2 U^{(f)}(x) = \left[|Dy|^2 \left(U(y) - \frac{1}{2}\mathscr{S}\left(f^{-1}\right)\right)\right]_{y=f(x)}$.

The difference of second order operators is a scalar operator:

$$\Delta^{(f)} - \Delta = \left[|Dy|^2 \left(U(y) - \frac{1}{2}\mathscr{S}\left(f^{-1}\right)\right)\right]_{y=f(x)} - |Dx|^2 U(x)$$

In particular if $U(x) = 0$ then

$$\Delta^{(f)} - \Delta = -\frac{1}{2}\mathscr{S}\left(f^{-1}\right)|Dy|^2 = -\frac{1}{2}\left(\frac{x_{yyy}}{x_y} - \frac{3}{2}\left(\frac{x_{yy}}{x_y}\right)^2\right)|Dy|^2$$

at $y = f(x)$.

Cobounbdary in a wider space=non-trivial cocycle in the space

Schwarzian derivative is a coboundary in the space of second order operators. This coboundary is an operator of zeroth order–it is an operator of multiplication on a density of weight 2.. It is a cocycle in the space of densities.

Coboundbdary in a wider space=non-trivial cocycle in the space

Schwarzian derivative is a coboundary in the space of second order operators. This coboundary is an operator of zeroth order–it is an operator of multiplication on a density of weight 2.. It is a cocycle in the space of densities.
This cocycle cannot be represented as a coboundary of density.

$$\underbrace{-\frac{1}{2}\left(\frac{x_{yyy}}{x_y}-\frac{3}{2}\left(\frac{x_{yy}}{x_y}\right)^2\right)|Dy|^2}_{\text{depends on 3-rd derivatives}}=\Delta^{(f)}-\Delta \neq \underbrace{S^{(f)}-S}_{\text{depends on derivatives }\leq 1}$$

Schwarzian — cohomology

Schwarzian derivative $\mathscr{S}(f)$ is a non-trivial cocycle of the group of diffeomorphisms of $\mathbf{RP}^1$ acting on the space of densitites of the weight 2, which vanishes on projective transformations. These conditions define the Schwarzian uniquely (up to a constant multiplier).

Schwarzian — cohomology

Schwarzian derivative $\mathcal{S}(f)$ is a non-trivial cocycle of the group of diffeomorphisms of $\mathbf{RP^1}$ acting on the space of densitites of the weight 2, which vanishes on projective transformations.
These conditions define the Schwarzian uniquely (up to a constant multiplier).

Theorem

$$H^1\left(\mathrm{Diff}(\mathbf{RP^1}), \mathcal{F}_\lambda\right) = \begin{cases} \mathbf{R} \text{ if } \lambda = 0, 1, 2 \\ 0 \text{ otherwise} \end{cases}$$

$\lambda = 0$, $c_0 = \log x_y$. It vanishes for Euclidean transformations $y = x + c$.

$\lambda = 1$, $c_1 = \frac{x_{yy}}{x_y}|Dy|$. It vanishes for affine transformations $y = ax + b$.

$\lambda = 2$, $c_2 = \left(\frac{x_{yyy}}{x_y} - \frac{3}{2}\left(\frac{x_{yy}}{x_y}\right)^2\right)|Dy|^2$. It vanishes for proj. transf. $y = \frac{ax+b}{cx+d}$.

Fine tuning of weights of densities

We come to beautiful results considering

$$\mathscr{F}_\lambda \xrightarrow{\Delta=\frac{\partial^2}{\partial x^2}} \mathscr{F}_\mu \text{ for } \lambda = -\frac{1}{2}, \mu = +\frac{3}{2}$$

$$\begin{cases} \mu - \lambda = 2 \\ \mu + \lambda = 1 \end{cases}$$

Fine tuning of weights of densities

We come to beautiful results considering

$$\mathcal{F}_\lambda \xrightarrow{\Delta=\frac{\partial^2}{\partial x^2}} \mathcal{F}_\mu \text{ for } \lambda = -\frac{1}{2}, \mu = +\frac{3}{2}$$

$$\begin{cases} \mu - \lambda = 2 \\ \mu + \lambda = 1 \end{cases} \quad \text{under an arbitrary diffeomorph.} \begin{cases} \frac{\partial^2}{\partial x^2} \to \frac{\partial^2}{\partial x^2} + \dots \\ \text{term} \dots \frac{\partial}{\partial x} \text{ does not appear} \end{cases}$$

Fine tuning of weights of densities

We come to beautiful results considering

$$\mathcal{F}_\lambda \xrightarrow{\Delta = \frac{\partial^2}{\partial x^2}} \mathcal{F}_\mu \text{ for } \lambda = -\frac{1}{2}, \mu = +\frac{3}{2}$$

$$\begin{cases} \mu - \lambda = 2 \\ \mu + \lambda = 1 \end{cases} \quad \text{under an arbitrary diffeomorph.} \begin{cases} \frac{\partial^2}{\partial x^2} \to \frac{\partial^2}{\partial x^2} + \dots \\ \text{term} \dots \frac{\partial}{\partial x} \text{ does not appear} \end{cases}$$

We come to $\lambda = -\frac{1}{2}, \mu = +\frac{3}{2}$.

Fine tuning of weights of densities

We come to beautiful results considering

$$\mathcal{F}_\lambda \xrightarrow{\Delta=\frac{\partial^2}{\partial x^2}} \mathcal{F}_\mu \text{ for } \lambda = -\frac{1}{2}, \mu = +\frac{3}{2}$$

$$\begin{cases} \mu - \lambda = 2 \\ \mu + \lambda = 1 \end{cases} \quad \text{under an arbitrary diffeomorph.} \begin{cases} \frac{\partial^2}{\partial x^2} \to \frac{\partial^2}{\partial x^2} + \dots \\ \text{term} \dots \frac{\partial}{\partial x} \text{ does not appear} \end{cases}$$

We come to $\lambda = -\frac{1}{2}, \mu = +\frac{3}{2}$.

Try to shine some light on these constructions.

Algebra of densities

Consider the space of all densities on a (super)manifold M:

$$\mathscr{F} = \oplus_{\lambda} \mathscr{F}_{\lambda}(M).$$

$$\mathscr{F} \ni \Psi = \Psi_{\lambda_1} |Dx|^{\lambda_1} + \Psi_{\lambda_2} |Dx|^{\lambda_2} + \cdots + \Psi_{\lambda_k} |Dx|^{\lambda_k}.$$

Algebra of densities

Consider the space of all densities on a (super)manifold M:

$$\mathscr{F} = \oplus_{\lambda} \mathscr{F}_{\lambda}(M)\,.$$

$$\mathscr{F} \ni \Psi = \Psi_{\lambda_1}|Dx|^{\lambda_1} + \Psi_{\lambda_2}|Dx|^{\lambda_2} + \cdots + \Psi_{\lambda_k}|Dx|^{\lambda_k}\,.$$

The vector space $\mathscr{F}(M)$ is a commutative algebra with respect to natural multiplication

$$\Psi_{\lambda_1}|Dx|^{\lambda_1} \cdot \Psi_{\lambda_2}|Dx|^{\lambda_2} = \Psi_{\lambda_1}\Psi_{\lambda_2}|Dx|^{\lambda_1+\lambda_2}\,.$$

Canonical scalar product in the algebra of densities.

Density $\Psi(x)|Dx|$ of the weight $\sigma = 1$ is an invariant object of integration over manifold: If under changing of coordinates $\Psi(x)|Dx| = \widetilde{\Psi}(x')|Dx'|$ then

$$\int \Psi(x)|Dx| = \int \widetilde{\Psi}(x')|Dx'|, \qquad \text{since } \widetilde{\Psi}(x') = \Psi(x(x')) \left|\det\left(\frac{\partial x(x')}{\partial x'}\right)\right|$$

We come to

Definition

Let $\mathbf{s}_1 = s_1(x)|Dx|^{\lambda_1}$ and $\mathbf{s}_2 = s_2(x)|Dx|^{\lambda_2}$ be two densities of weights λ_1, λ_2. Then

$$\langle \mathbf{s}_1, \mathbf{s}_1 \rangle = \begin{cases} \int s_1(x)s_2(x)|Dx| & \text{if } \lambda_1 + \lambda_2 = 1 \\ 0 & \text{if } \lambda_1 + \lambda_2 \neq 1 \end{cases}.$$

Useful symbolic notation

$s(x)|Dx|^{\lambda} \leftrightarrow s(x)t^{\lambda}$.
Density $\Psi(x,t) = \sum \Psi_k(x)t^{\lambda_k} \leftrightarrow \sum \Psi_k(x)|Dx|^{\lambda_k}$

$$\langle a(x,t), b(x,t)\rangle = \int_M \mathrm{Res}\left(\frac{a(x,t)b(x,t)}{t^2}\right) Dx\,.$$

Differential operators on densities

Differential operators $D = D(x, t, \frac{\partial}{\partial x}, \frac{\partial}{\partial t})$ act on algebra $\mathscr{F}$ of densities.

Examples.

Weight operator: $\hat{w} = t\frac{\partial}{\partial t}$. $t\frac{\partial}{\partial t}\left(a(x)t^\lambda\right) = \lambda a(x)t^\lambda$.

Differentiation of algebra $\mathscr{F}$.

$$\hat{A} = t^\delta \left(A^a(x)\partial_a + A_0\hat{w}\right)$$. (Vector field of the weight δ).

$$\hat{A}\left(\Psi(x)|Dx|^\lambda\right) = t^\delta\left((A^a(x)\partial_a + A_0\hat{w})\left(\Psi t^\lambda\right) = t^{\lambda+\delta}\left(A^a\partial_a\Psi + \lambda A_0\Psi\right)\right.$$

Let $\mathbf{X}$ be a vector field on M:

$$L_{\mathbf{X}}\left(\Psi|Dx|^\lambda\right) = \left(X^a\partial_a\Psi + \lambda\partial_a X^a\Psi\right).$$

It defines the vector field

$$\mathscr{L}_{\mathbf{X}} = X^a\partial_a + \partial_a X^a t\frac{\partial}{\partial t} = X^a\partial_a + \partial_a X^a\hat{w} \qquad \text{on algebra } \mathscr{F}.$$

Self-adjoint operators

Examples of adjoints

$\partial_a^+ = -\partial_a, t^+ = t, \left(\frac{\partial}{\partial t}\right)^+ = -\frac{\partial}{\partial t} + \frac{2}{t}, \hat{w}^+ = 1 - \hat{w}.$

n-th order operator A is self-adjoint (anti-self-adjoint) if

$$A^+ = (-1)^n A\,.$$

Self-adjoint operators

Examples of adjoints

$\partial_a^+ = -\partial_a, t^+ = t, \left(\frac{\partial}{\partial t}\right)^+ = -\frac{\partial}{\partial t} + \frac{2}{t}, \hat{w}^+ = 1 - \hat{w}.$

n-th order operator A is self-adjoint (anti-self-adjoint) if

$$A^+ = (-1)^n A\,.$$

Example: Lie derivative anti-self-adjoint operator:

$$(\mathscr{L}_{\mathbf{X}})^+ = \left(X^a \partial_a + \partial_a X^a \hat{w}\right) = -\partial_a X^a - X^a \partial_a + (1-\hat{w})\partial_a X^a = -\mathscr{L}_{\mathbf{X}}\,.$$

Self-adjoint operators

Examples of adjoints

$\partial_a^+ = -\partial_a, t^+ = t, \left(\frac{\partial}{\partial t}\right)^+ = -\frac{\partial}{\partial t} + \frac{2}{t}, \hat{w}^+ = 1 - \hat{w}$.

n-th order operator A is self-adjoint (anti-self-adjoint) if

$$A^+ = (-1)^n A\,.$$

Example: Lie derivative anti-self-adjoint operator:

$$(\mathscr{L}_{\mathbf{X}})^+ = \left(X^a\partial_a + \partial_a X^a \hat{w}\right) = -\partial_a X^a - X^a\partial_a + (1-\hat{w})\partial_a X^a = -\mathscr{L}_{\mathbf{X}}\,.$$

This means that $\mathscr{L}_{\mathbf{X}}$ is divergence-less field.
$\mathrm{div}\,\hat{A} = -(\hat{A} + \hat{A^+}) = t^\delta\left(\partial_a A^a + (\delta - 1)A_0\right)$ for vector field
$\hat{A} = t^\delta\left(A^a\partial_a + A_0\hat{w}\right)$. $\mathrm{div}\,\mathscr{L}_{\mathbf{X}} = 0$.

$n = 1$. First order operators on $\mathscr{F}(M)$.

Simple but important observation:
Let M be an arbitrary (orientable) (super)manifold.
Anti-self-adjoint first order operator of the weight δ on algebra of densities $\mathscr{F}(M)$ has the following appearance

$$\hat{A} = t^{\delta}\left(A^a\partial_a + \frac{\partial_a A^a \hat{w}}{1-\delta}\right).$$

$n=1$. First order operators on $\mathscr{F}(M)$.

Simple but important observation:
Let M be an arbitrary (orientable) (super)manifold.
Anti-self-adjoint first order operator of the weight δ on algebra of densities $\mathscr{F}(M)$ has the following appearance

$$\hat{A}=t^{\delta}\left(A^a\partial_a+\frac{\partial_a A^a\hat{w}}{1-\delta}\right).$$

It defines the pencil $\{\hat{A}_\lambda\}$ of operators on spaces $\mathscr{F}_\lambda$:

$$\hat{A}_\lambda=\hat{A}\big|_{\mathscr{F}_\lambda}=\hat{A}=t^{\delta}\left(A^a\partial_a+\frac{\lambda\partial_a A^a}{1-\delta}\right),\ \lambda\in\mathbf{R}..$$

If $\delta=0$ then the operators $\hat{A}_\lambda$ of this pencil are just usual Lie derivatives of densities of weight λ:

$$\hat{A}_\lambda=L_{\mathbf{A}}\big|_{\mathscr{F}_\lambda}=A^a\partial_a+\lambda\partial_a A^a.$$

Two important cases

We consider two examples.

1. Self-adjoint second order operator on algebra of densities $\mathscr{F}(M)$ on an arbitrary (super)manifold M and corresponding pencil of second order operators on spaces $\mathscr{F}_\lambda(M)$.
(T.T.Voronov, H.M.Kh., 2003.).

2. (Anti)-self-adjoint n-th order operator on algebra of densities $\mathscr{F}(\mathbf{R})$ on the real line $\mathbf{R}$ and corresponding pencil of second order operators on spaces $\mathscr{F}_\lambda(\mathbf{R})$.
(A.M.Biggs, H.M.Kh., 2011.).

Two important cases

We consider two examples.

1. Self-adjoint second order operator on algebra of densities $\mathscr{F}(M)$ on an arbitrary (super)manifold M and corresponding pencil of second order operators on spaces $\mathscr{F}_\lambda(M)$.
(T.T.Voronov, H.M.Kh., 2003.).

2. (Anti)-self-adjoint n-th order operator on algebra of densities $\mathscr{F}(\mathbf{R})$ on the real line $\mathbf{R}$ and corresponding pencil of second order operators on spaces $\mathscr{F}_\lambda(\mathbf{R})$.
(A.M.Biggs, H.M.Kh., 2011.).

This is a principal result

Recalling: connection on densities

$$\nabla:\ \nabla_{\mathbf{X}}(\Psi|Dx|^{\lambda}) = \partial_{\mathbf{X}}\Psi|Dx|^{\lambda} + \lambda\Psi|Dx|^{\lambda-1}\partial_{\mathbf{X}}|Dx| = X^{a}(\partial_a\Psi + \lambda\Gamma_a\Psi|Dx|^{\lambda}$$

Γ_a are "Cristoffel" symbols of connection: $\Gamma_a|Dx| = \nabla_{\partial_a}|Dx|$.

Under changing of coordinates $x^{a} = x^{a}(x^{a'})$

$$\Gamma_{a'} = \frac{\partial x^{a}}{\partial x^{a'}}\left(\Gamma_a + \partial_a\left(\left|\frac{Dx'}{Dx}\right|\right)\right), \qquad \left|\frac{Dx'}{Dx}\right| = \det\left(\frac{\partial x^{a'}}{\partial x^{a}}\right).$$

Recalling: connection on densities

$$\nabla\colon \nabla_{\mathbf{X}}(\Psi|Dx|^{\lambda}) = \partial_{\mathbf{X}}\Psi|Dx|^{\lambda} + \lambda\Psi|Dx|^{\lambda-1}\partial_{\mathbf{X}}|Dx| = X^{a}(\partial_a\Psi + \lambda\Gamma_a\Psi|Dx|^{\lambda}$$

Γ_a are "Cristoffel" symbols of connection: $\Gamma_a|Dx| = \nabla_{\partial_a}|Dx|$.

Under changing of coordinates $x^a = x^a(x^{a'})$

$$\Gamma_{a'} = \frac{\partial x^a}{\partial x^{a'}}\left(\Gamma_a + \partial_a\left(\left|\frac{Dx'}{Dx}\right|\right)\right), \qquad \left|\frac{Dx'}{Dx}\right| = \det\left(\frac{\partial x^{a'}}{\partial x^a}\right).$$

Basic examples of connection

1. An arbitrary volume form $\mathbf{s} = \rho(x)|Dx|$ defines a connection

$$\nabla^{(\rho)}_{\mathbf{X}}\left(\Psi(x)|Dx|^{\lambda}\right) = \mathbf{s}^{\lambda}\partial_{\mathbf{X}}\left(\frac{\Psi(x)|Dx|^{\lambda}}{\mathbf{s}^{\lambda}}\right) = X^{a}\left(\partial_a\Psi + \Gamma_a\psi\right),$$

with $\Gamma_a = -\partial_a\log\rho$.

2. An arbitrary affine connection with Christoffel symbols $\{\Gamma^a_{bc}\}$ define connection on densitites ∇ such that $\nabla_{\partial_a}|Dx| = \Gamma_a|Dx|$ with $\Gamma = -\Gamma^b_{ab}$.

Second order self-adjoint operator on $\mathscr{F}(M)$

.

Theorem

Let Δ be a second order operator on $\mathscr{F}(M)$ of the weight δ such that $\Delta^+ = \Delta$ and $\Delta 1 = 0$. Then

$$\Delta = \frac{t^\delta}{2}\left(S^{ab}\partial_b\partial_a + \left(\partial_b S^{ba}(-1)^{p(b)(p(\varepsilon)+1)}(2\hat{w}+\delta-1)\Gamma^a\right)\partial_a\right)$$

$$+\frac{t^\delta}{2}\left(\hat{w}\partial_a\Gamma^a(-1)^{p(a)(p(\varepsilon)+1)} + \hat{w}(\hat{w}+\delta-1)\theta\right),$$ *(T.Voronov, H.Kh., 2003)*

Second order self-adjoint operator on $\mathscr{F}(M)$

.

Theorem

Let Δ be a second order operator on $\mathscr{F}(M)$ of the weight δ such that $\Delta^+ = \Delta$ and $\Delta 1 = 0$. Then

$$\Delta = \frac{t^\delta}{2}\left(S^{ab}\partial_b\partial_a + \left(\partial_b S^{ba}(-1)^{p(b)(p(\varepsilon)+1)}(2\hat{w}+\delta-1)\Gamma^a\right)\partial_a\right)$$

$$+\frac{t^\delta}{2}\left(\hat{w}\partial_a\Gamma^a(-1)^{p(a)(p(\varepsilon)+1)}+\hat{w}(\hat{w}+\delta-1)\theta\right),$$ *(T.Voronov, H.Kh., 2003)*

where $p(a)$ is a parity of coordinate x^a, $p(\varepsilon)$ a parity of Δ,

Second order self-adjoint operator on $\mathscr{F}(M)$

.

Theorem

Let Δ be a second order operator on $\mathscr{F}(M)$ of the weight δ such that $\Delta^+ = \Delta$ and $\Delta 1 = 0$. Then

$$\Delta = \frac{t^\delta}{2}\left(S^{ab}\partial_b\partial_a + \left(\partial_b S^{ba}(-1)^{p(b)(p(\varepsilon)+1)}(2\hat{w}+\delta-1)\Gamma^a\right)\partial_a\right)$$

$$+\frac{t^\delta}{2}\left(\hat{w}\partial_a\Gamma^a(-1)^{p(a)(p(\varepsilon)+1)} + \hat{w}(\hat{w}+\delta-1)\theta\right),$$ *(T.Voronov, H.Kh., 2003)*

where $p(a)$ is a parity of coordinate x^a, $p(\varepsilon)$ a parity of Δ,
$S^{ab}(x)t^\delta = S^{ab}(x)|Dx|^\delta$ is symmetric tensor field $\otimes$ density of weight δ,

Second order self-adjoint operator on $\mathscr{F}(M)$

.

Theorem

Let Δ be a second order operator on $\mathscr{F}(M)$ of the weight δ such that $\Delta^+ = \Delta$ and $\Delta 1 = 0$. Then

$$\Delta = \frac{t^\delta}{2}\left(S^{ab}\partial_b\partial_a + \left(\partial_b S^{ba}(-1)^{p(b)(p(\varepsilon)+1)}(2\hat{w}+\delta-1)\Gamma^a\right)\partial_a\right)$$

$$+\frac{t^\delta}{2}\left(\hat{w}\partial_a\Gamma^a(-1)^{p(a)(p(\varepsilon)+1)} + \hat{w}(\hat{w}+\delta-1)\theta\right)$$, *(T.Voronov, H.Kh., 2003)*

where $p(a)$ is a parity of coordinate x^a, $p(\varepsilon)$ a parity of Δ,
$S^{ab}(x)t^\delta = S^{ab}(x)|Dx|^\delta$ is symmetric tensor field $\otimes$ density of weight δ,
$\Gamma^a|Dx|^\delta$ is an upper connection $\otimes$ density of weight δ It transform as
$S^{ab}(x)|Dx|^\delta\Gamma_b$

Second order self-adjoint operator on $\mathscr{F}(M)$

.

Theorem

Let Δ be a second order operator on $\mathscr{F}(M)$ of the weight δ such that $\Delta^+ = \Delta$ and $\Delta 1 = 0$. Then

$$\Delta = \frac{t^\delta}{2}\left(S^{ab}\partial_b\partial_a + \left(\partial_b S^{ba}(-1)^{p(b)(p(\varepsilon)+1)}(2\hat{w}+\delta-1)\Gamma^a\right)\partial_a\right)$$

$$+\frac{t^\delta}{2}\left(\hat{w}\partial_a\Gamma^a(-1)^{p(a)(p(\varepsilon)+1)} + \hat{w}(\hat{w}+\delta-1)\theta\right), \text{(T.Voronov, H.Kh., 2003)}$$

where $p(a)$ is a parity of coordinate x^a, $p(\varepsilon)$ a parity of Δ,
$S^{ab}(x)t^\delta = S^{ab}(x)|Dx|^\delta$ is symmetric tensor field $\otimes$ density of weight δ,
$\Gamma^a|Dx|^\delta$ is an upper connection $\otimes$ density of weight δ It transform as
$S^{ab}(x)|Dx|^\delta\Gamma_b$
θ is a Brans-Dicke type "scalar" density. It transforms as $\Gamma_a S^{ab}(x)|Dx|^\delta\Gamma_b$.

The special case

Consider the above operator in the case if the weight $\delta = 0$ and the principal symbol S^{ab} is invertible. Then

$$\Delta = \frac{1}{2}\left(S^{ab}\partial_b\partial_a + \left(\partial_b S^{ba} + (2\hat{w}-1)\Gamma^a\right)\partial_a + \hat{w}\partial_a\Gamma^a + \hat{w}(\hat{w}-1)\theta\right)$$

where Γ_a is a connection on densities,($S^{ab}\Gamma_b = \Gamma^a$) and $\theta = \Gamma^a\Gamma_a$ (we omit here terms $(-1)^{\cdots}$)

Pencil of Laplacians on $\mathcal{F}_\lambda(M)$

The above operator defines the pencil: $\lambda \in \mathbf{R}$, $\Delta_\lambda = \Delta\big|_{\mathcal{F}_\lambda} =$

$$\frac{1}{2}\left(S^{ab}\partial_b\partial_a + \left(\partial_b S^{ba} + (2\lambda - 1)\Gamma^a\right)\partial_a + \lambda\partial_a\Gamma^a + \lambda(\lambda-1)\Gamma^a\Gamma_a\right)$$

Remark The pencil possesses the singular points, the weights $\lambda = 0, \frac{1}{2}, 1$. The weight $\lambda = \frac{1}{2}$ is of the most interest.

n-th order operator on densities on $\mathbf{R}$

.

Proposition. Let L be n-th order operator of the weight δ on the algebra $\mathscr{F}(\mathbf{R})$ such that $L^+ = (-1)^n L$. Then $L = t^\delta s \frac{\partial}{\partial x^n} +$

$$t^\delta \left(\frac{n}{2} \left(s_x + 2s\Gamma \hat{w}_{n+\delta} \right) \frac{\partial}{\partial x^{n-1}} + \frac{n(n-1)}{2} \left((s\Gamma)_x + s(\Gamma^2 + \tau)\hat{w}_{n+\delta} \right) \hat{w}_{n+\delta} \frac{\partial}{\partial x^{n-2}} + \right)$$

$$t^\delta \frac{n(n-1)}{2} \left(\frac{n-2}{6} s_x \Gamma - \frac{n+1+3\delta}{6} s\Gamma_x - \frac{n+1+3\delta(\delta+1)}{12} s\Gamma^2 + s\sigma \right) \frac{\partial}{\partial x^{n-2}} + \dots,$$

where $\hat{w}_s = \hat{w} + \frac{s-1}{2}$. Here $s = s(x)|Dx|^{\delta-n}$ is a density of weight $\delta - n$,

$\tau(x)|Dx|^2, \sigma(x)|Dx|^2$ are densities of weight 2,

Γ is a connection

(In a dimension 1, $\Gamma = -\partial_x(\log \rho)$ for a volume form $\rho(x)dx$)

(A.Biggs, H,Kh. (2011).)

Special case $\delta = n$

We put $s = 1$, $\tau = \sigma = 0$ and come to

$$K_n = t^n \left(\frac{\partial}{\partial x^n} + n\Gamma \hat{w}_n \frac{\partial}{\partial x^{n-1}} + \frac{n(n-1)}{2} \left(\Gamma_x + \Gamma^2 \hat{w}_n \right) \hat{w}_n \frac{\partial}{\partial x^{n-2}} + \right)$$

$$-t^n \frac{n(n-1)(n+1)}{12} \left(\Gamma_x + \frac{1}{2}\Gamma^2 \right) \frac{\partial}{\partial x^{n-2}} + \dots$$

where $\hat{w}_n = \hat{w} + \frac{n-1}{2}$, $\hat{w}_n|_{\mathscr{F}_\lambda} = \lambda + \frac{n-1}{2}$:

$$\hat{w}_n \left(\Psi(x)|Dx|^\lambda \right) = \left(t\frac{\partial}{\partial t} + \frac{n-1}{2} \right) \left(\Psi(x) t^\lambda \right) = \left(\lambda + \frac{n-1}{2} \right) \Psi(x)|Dx|^\lambda .$$

Example of n-th order operator

Consider $D = |Dx|\left(\frac{\partial}{\partial x} + \hat{w}\Gamma\right)$. It is anti-self-adjoint operator[1] :

$$D^+ = \left(t\left(\frac{\partial}{\partial x} + \hat{w}\Gamma\right)\right)^+ = -\left(t\left(\frac{\partial}{\partial x} + \hat{w}\Gamma\right)\right), \qquad (t = |Dx|).$$

[1] It is similar to de Rham differential: $D|_{\mathscr{F}_\lambda} = \frac{1}{\mathbf{s}^\lambda}|Dx|\frac{\partial}{\partial x}\mathbf{s}^\lambda$, where $\mathbf{s} = \rho(x)|Dx|$ is such that $\Gamma = -\partial_x(\log\rho)$.

Example of *n*-th order operator

Consider $D=|Dx|\left(\frac{\partial}{\partial x}+\hat{w}\Gamma\right)$. It is anti-self-adjoint operator[1] :

$$D^+=\left(t\left(\frac{\partial}{\partial x}+\hat{w}\Gamma\right)\right)^+=-\left(t\left(\frac{\partial}{\partial x}+\hat{w}\Gamma\right)\right), \qquad (t=|Dx|).$$

$$D^n==t^n\left(\frac{\partial}{\partial x^n}+n\Gamma\hat{w}_n\frac{\partial}{\partial x^{n-1}}+\frac{n(n-1)}{2}\left(\Gamma_x+\Gamma^2\hat{w}_n\right)\hat{w}_n\frac{\partial}{\partial x^{n-2}}+\right)$$
$$-t^n\frac{n(n-1)(n+1)}{12}\left(\Gamma_x+\frac{1}{2}\Gamma^2\right)\frac{\partial}{\partial x^{n-2}}+\dots$$

[1] It is similar to de Rham differential: $D|_{\mathscr{F}_\lambda}=\frac{1}{\mathbf{s}^\lambda}|Dx|\frac{\partial}{\partial x}\mathbf{s}^\lambda$, where $\mathbf{s}=\rho(x)|Dx|$ is such that $\Gamma=-\partial_x(\log\rho)$.

Pencil of operators on $\mathscr{F}(\mathbf{R})$

The operator K_n defines the pencil:

$$\lambda \in \mathbf{R}, \Delta_\lambda = \Delta\big|_{\mathscr{F}_\lambda} = \Delta\left(\hat{w}_n = \hat{w} + \frac{n-1}{2} \to \lambda_n = \lambda + \frac{n-1}{2}\right) =$$

$$t^n \left(\frac{\partial}{\partial x^n} + n\Gamma\lambda_n \frac{\partial}{\partial x^{n-1}} + \frac{n(n-1)}{2}\left(\Gamma_x + \Gamma^2\lambda_n\right)\lambda_n \frac{\partial}{\partial x^{n-2}} + \right)$$

$$-t^n \frac{n(n-1)(n+1)}{12}\left(\Gamma_x + \frac{1}{2}\Gamma^2\right)\frac{\partial}{\partial x^{n-2}} + \dots$$

Pencil of operators on $\mathscr{F}(\mathbf{R})$

The operator K_n defines the pencil:

$$\lambda \in \mathbf{R}, \Delta_\lambda = \Delta\big|_{\mathscr{F}_\lambda} = \Delta\left(\hat{w}_n = \hat{w} + \frac{n-1}{2} \to \lambda_n = \lambda + \frac{n-1}{2}\right) =$$

$$t^n\left(\frac{\partial}{\partial x^n} + n\Gamma\lambda_n \frac{\partial}{\partial x^{n-1}} + \frac{n(n-1)}{2}\left(\Gamma_x + \Gamma^2\lambda_n\right)\lambda_n \frac{\partial}{\partial x^{n-2}} + \right)$$

$$-t^n \frac{n(n-1)(n+1)}{12}\left(\Gamma_x + \frac{1}{2}\Gamma^2\right)\frac{\partial}{\partial x^{n-2}} + \dots$$

This pencil possesses a singular point $\lambda_n = 0$, i.e. $\lambda = \frac{1-n}{2}$.

Pencil of operators on $\mathscr{F}(\mathbf{R})$: its singular point

The operator K_n defines the pencil:

$$\lambda \in \mathbf{R}, \Delta_\lambda = \Delta\Big|_{\mathscr{F}_\lambda} = \Delta\left(\hat{w}_n = \hat{w} + \frac{n-1}{2} \to \lambda_n = \lambda + \frac{n-1}{2}\right) =$$

$$t^n\left(\frac{\partial}{\partial x^n} + n\Gamma\lambda_n\frac{\partial}{\partial x^{n-1}} + \frac{n(n-1)}{2}\left(\Gamma_x + \Gamma^2\lambda_n\right)\lambda_n\frac{\partial}{\partial x^{n-2}} + \right)$$

$$-t^n\frac{n(n-1)(n+1)}{12}\left(\Gamma_x + \frac{1}{2}\Gamma^2\right)\frac{\partial}{\partial x^{n-2}} + \dots$$

This pencil possesses a singular point $\lambda_n = 0$, i.e. $\lambda = \frac{1-n}{2}$.

Singular point of the pencil: $\lambda = \frac{1-n}{2}$

$$\lambda_n = 0 \Rightarrow \lambda = \frac{1-n}{2}, \quad \Delta_{\frac{1-n}{2}} : \mathscr{F}_{\frac{1-n}{2}} \to \mathscr{F}_{\frac{1-n}{2}+n} = \mathscr{F}_{\frac{1+n}{2}}$$

$$\Delta_{\frac{1-n}{2}} = t^n \left(\frac{\partial}{\partial x^n} - \frac{n(n-1)(n+1)}{12} \left(\Gamma_x + \frac{1}{2}\Gamma^2 \right) \frac{\partial}{\partial x^{n-2}} + \dots \right)$$

Singular point of the pencil: $\lambda = \frac{1-n}{2}$

$$\lambda_n = 0 \Rightarrow \lambda = \frac{1-n}{2}, \quad \Delta_{\frac{1-n}{2}} : \mathscr{F}_{\frac{1-n}{2}} \to \mathscr{F}_{\frac{1-n}{2}+n} = \mathscr{F}_{\frac{1+n}{2}}$$

$$\Delta_{\frac{1-n}{2}} = t^n \left(\frac{\partial}{\partial x^n} - \frac{n(n-1)(n+1)}{12} \left(\Gamma_x + \frac{1}{2}\Gamma^2 \right) \frac{\partial}{\partial x^{n-2}} + \dots \right)$$

For $n = 2$

$$\Delta_{-\frac{1}{2}} : \mathscr{F}_{-\frac{1}{2}} \to \mathscr{F}_{\frac{3}{2}}$$

Singular point of the pencil: $\lambda = \frac{1-n}{2}$

$$\lambda_n = 0 \Rightarrow \lambda = \frac{1-n}{2}, \quad \Delta_{\frac{1-n}{2}} : \mathcal{F}_{\frac{1-n}{2}} \to \mathcal{F}_{\frac{1-n}{2}+n} = \mathcal{F}_{\frac{1+n}{2}}$$

$$\Delta_{\frac{1-n}{2}} = t^n \left(\frac{\partial}{\partial x^n} - \frac{n(n-1)(n+1)}{12} \left(\Gamma_x + \frac{1}{2}\Gamma^2 \right) \frac{\partial}{\partial x^{n-2}} + \dots \right)$$

For $n = 2$

$$\Delta_{-\frac{1}{2}} : \mathcal{F}_{-\frac{1}{2}} \to \mathcal{F}_{\frac{3}{2}}$$

$$\Delta_{-\frac{1}{2}} = t^2 \left(\frac{\partial}{\partial x^2} - \frac{1}{2} \left(\Gamma_x + \frac{1}{2}\Gamma^2 \right) \right) = |Dx|^2 \left(\frac{\partial}{\partial x^2} - \frac{1}{2} \left(\Gamma_x + \frac{1}{2}\Gamma^2 \right) \right)$$

Sturm-Liouville operator

Compare it with the latter Sturm-Liouville operator

$$\Delta = |Dx|^2 \left(\frac{\partial^2}{\partial x^2} + U(x) \right) = |Dx|^2 \left(\frac{\partial}{\partial x^2} - \frac{1}{2} \left(\Gamma_x + \frac{1}{2} \Gamma^2 \right) \right).$$

$$\Delta : \quad \mathscr{F}_{-\frac{1}{2}} \to \mathscr{F}_{\frac{3}{2}}.$$

Sturm-Liouville operator

Compare it with the latter Sturm-Liouville operator

$$\Delta = |Dx|^2 \left(\frac{\partial^2}{\partial x^2} + U(x) \right) = |Dx|^2 \left(\frac{\partial}{\partial x^2} - \frac{1}{2} \left(\Gamma_x + \frac{1}{2} \Gamma^2 \right) \right) .$$

$\Delta: \quad \mathcal{F}_{-\frac{1}{2}} \to \mathcal{F}_{\frac{3}{2}}$. Potential $U(x)|Dx|^2$ is a function of connection

$$-U(x)|Dx|^2 = \mathbf{U}_\Gamma = \frac{1}{2} \left(\Gamma_x + \frac{1}{2} \Gamma^2 \right) |Dx|^2 .$$

Sturm-Liouville operator

Compare it with the latter Sturm-Liouville operator

$$\Delta = |Dx|^2 \left(\frac{\partial^2}{\partial x^2} + U(x)\right) = |Dx|^2 \left(\frac{\partial}{\partial x^2} - \frac{1}{2}\left(\Gamma_x + \frac{1}{2}\Gamma^2\right)\right).$$

$\Delta: \quad \mathscr{F}_{-\frac{1}{2}} \to \mathscr{F}_{\frac{3}{2}}$. Potential $U(x)|Dx|^2$ is a function of connection

$$-U(x)|Dx|^2 = \mathbf{U}_\Gamma = \frac{1}{2}\left(\Gamma_x + \frac{1}{2}\Gamma^2\right)|Dx|^2.$$

Remark Since $n = 1$ one can always choose a volume form ρ such that $\Gamma = -\partial_x \log \rho$. Respectively one can always choose a coordinate x such that $\rho = 1$, $\Gamma = 0$ and $\mathbf{U}_\Gamma = 0$ in this coordinate.

Variation of connection, "potential" U, and Δ under diffeomorphism

For diffeomorphism $f = y(x)$

$$\Gamma^{(f)}(x)|Dx| = \Gamma(y)\,|Dy|\big|_{y=y(x)} + y_x \partial_y \log x_y\,|Dx|\,,$$

respectively

$$\Delta^{(f)} - \Delta = -\frac{1}{2}\left(\mathbf{U}_{\Gamma^{(f)}} - \mathbf{U}_{\Gamma}\right) =$$

$$U_\Gamma(y(x))|Dy|^2 - U_\Gamma(x)|Dx|^2 + \underbrace{\left(\frac{x_{yyy}}{x_y} - \frac{3}{2}\frac{x_{yy}^2}{x_y^2}\right)|Dy|^2}_{\text{Schwarzian } \mathscr{S}(f^{-1})}$$

at $y = y(x)$.

Variation of connection, "potential" U, and Δ under diffeomorphism

For diffeomorphism $f = y(x)$

$$\Gamma^{(f)}(x)|Dx| = \Gamma(y)\,|Dy|\big|_{y=y(x)} + y_x \partial_y \log x_y\,|Dx|\,,$$

respectively

$$\Delta^{(f)} - \Delta = -\frac{1}{2}\left(\mathbf{U}_{\Gamma^{(f)}} - \mathbf{U}_\Gamma\right) =$$

$$U_\Gamma(y(x))|Dy|^2 - U_\Gamma(x)|Dx|^2 + \underbrace{\left(\frac{x_{yyy}}{x_y} - \frac{3}{2}\frac{x_{yy}^2}{x_y^2}\right)|Dy|^2}_{\text{Schwarzian } \mathscr{S}(f^{-1})}$$

at $y = y(x)$.

If in coordinate x, $\Gamma = 0$, then $U = 0$ and $\Delta^{(f)} - \Delta = \mathscr{S}(f^{-1})|Dy|^2$.

Variation of potential under changing of connection

Difference of connections is a (co)vector $\Gamma' - \Gamma = \mathbf{X}$.

For $n = 1$ $\Gamma'(x)|Dx| - \Gamma(x)|Dx| = X|Dx|$.

Variation of potential under changing of connection

Difference of connections is a (co)vector $\Gamma' - \Gamma = \mathbf{X}$.

For $n = 1$ $\Gamma'(x)|Dx| - \Gamma(x)|Dx| = X|Dx|$. $\mathbf{U}_{\Gamma^f} - \mathbf{U}_\Gamma =$

$$\frac{1}{2}\left(\Gamma'_x + \frac{1}{2}(\Gamma')^2\right) - \frac{1}{2}\left(\Gamma_x + \frac{1}{2}\Gamma^2\right)$$

Variation of potential under changing of connection

Difference of connections is a (co)vector $\Gamma' - \Gamma = \mathbf{X}$.

For $n = 1$ $\Gamma'(x)|Dx| - \Gamma(x)|Dx| = X|Dx|$. $\mathbf{U}_{\Gamma^f} - \mathbf{U}_\Gamma =$

$$\frac{1}{2}\left(\Gamma'_x + \frac{1}{2}(\Gamma')^2\right) - \frac{1}{2}\left(\Gamma_x + \frac{1}{2}\Gamma^2\right) = \frac{1}{2}(\Gamma + X)_x + \frac{1}{4}(\Gamma + X)^2 -$$

$$\frac{1}{2}\left(\Gamma_x + \frac{1}{2}\Gamma^2\right) = \frac{1}{2}(X_x + \Gamma X) + \frac{1}{4}X^2 =$$

Variation of potential under changing of connection

Difference of connections is a (co)vector $\Gamma' - \Gamma = \mathbf{X}$.

For $n = 1$ $\Gamma'(x)|Dx| - \Gamma(x)|Dx| = X|Dx|$. $\mathbf{U}_{\Gamma^f} - \mathbf{U}_\Gamma =$

$$\frac{1}{2}\left(\Gamma'_x + \frac{1}{2}(\Gamma')^2\right) - \frac{1}{2}\left(\Gamma_x + \frac{1}{2}\Gamma^2\right) = \frac{1}{2}(\Gamma + X)_x + \frac{1}{4}(\Gamma + X)^2 -$$

$$\frac{1}{2}\left(\Gamma_x + \frac{1}{2}\Gamma^2\right) = \frac{1}{2}(X_x + \Gamma X) + \frac{1}{4}X^2 = \frac{1}{2}\left(\mathrm{div}\,\mathbf{X} + \frac{1}{2}\mathbf{X}^2\right).$$

Variation of potential under changing of connection

Difference of connections is a (co)vector $\Gamma' - \Gamma = \mathbf{X}$.
For $n = 1$ $\Gamma'(x)|Dx| - \Gamma(x)|Dx| = X|Dx|$. $\mathbf{U}_{\Gamma^f} - \mathbf{U}_\Gamma =$

$$\frac{1}{2}\left(\Gamma'_x + \frac{1}{2}(\Gamma')^2\right) - \frac{1}{2}\left(\Gamma_x + \frac{1}{2}\Gamma^2\right) = \frac{1}{2}(\Gamma + X)_x + \frac{1}{4}(\Gamma + X)^2 -$$

$$\frac{1}{2}\left(\Gamma_x + \frac{1}{2}\Gamma^2\right) = \frac{1}{2}(X_x + \Gamma X) + \frac{1}{4}X^2 = \frac{1}{2}\left(\mathrm{div}\,\mathbf{X} + \frac{1}{2}\mathbf{X}^2\right).$$

Changing of connection $\approx$ diffeomorphism:

$$\Gamma' = \Gamma + \mathbf{X} \leftrightarrow \exists f : \Gamma' - \Gamma^{(f)}$$

$$\mathrm{div}\,\mathbf{X} + \frac{1}{2}\mathbf{X}^2 = \left(U_\Gamma(y)|Dy| + \mathscr{S}(f^{-1})|Dy|\right)\Big|_{y=f(x)} - U_\Gamma(x)|Dx|.$$

Calculations.

Let x be a coordinate such that $\Gamma = 0$ in this coordinate. If $\Gamma' = \Gamma^{(f)}$, where $f = y(x)$ then

$$\Gamma'|Dx| = X|Dx| = y_x \partial_y \log x_y = -\partial_x \log y_x |Dx|, X = -\partial_x \log y_x \,.$$

$$\mathrm{div}\,\mathbf{X} + \frac{1}{2}\mathbf{X}^2 = 0 \leftrightarrow \frac{\partial X(x)}{\partial x} + \frac{1}{2}X^2 = 0 \leftrightarrow X(x) = \frac{2}{C+x}\,.$$

$$X(x) = -\partial_x \log y_x = \frac{2}{C+x} \leftrightarrow y_x = \frac{K}{(c+x)^2}$$

$$y(x) = \frac{ax+b}{cx+d} \text{ projective transformation.}$$

Laplacian on semidensity

Return to the pencil of second order operators on arbitrary supermanifold

$$\Delta_\lambda = \frac{1}{2}\left(S^{ab}\partial_b\partial_a + \left(\partial_b S^{ba} + (2\lambda - 1)\Gamma^a\right)\partial_a + \lambda\partial_a\Gamma^a + \lambda(\lambda - 1)\Gamma^a\Gamma_a\right)$$

Laplacian on semidensity

Return to the pencil of second order operators on arbitrary supermanifold

$$\Delta_\lambda = \frac{1}{2}\left(S^{ab}\partial_b\partial_a + \left(\partial_b S^{ba} + (2\lambda - 1)\Gamma^a\right)\partial_a + \lambda\partial_a\Gamma^a + \lambda(\lambda - 1)\Gamma^a\Gamma_a\right)$$

For the singular point $\lambda = \frac{1}{2}$ we have a laplacian

$$\Delta_{1/2} = \frac{1}{2}\left(S^{ab}\partial_b\partial_a + \partial_b S^{ba}\partial_a + \frac{1}{2}\partial_a\Gamma^a - \frac{1}{4}\Gamma^a\Gamma_a\right)$$

acting on semidensities $\Delta_{1/2}\colon\ \mathscr{F}_{1/2}(M) \to \mathscr{F}_{1/2}(M)$.

Laplacian on semidensities

Let M be an odd symplectic supermanifold equipped with non-degenerate Poisson bracket (anti-bracket) $\Omega^{ab}\colon \{z^a, z^b\} = \Omega^{ab}$.

Laplacian on semidensities

Let M be an odd symplectic supermanifold equipped with non-degenerate Poisson bracket (anti-bracket) Ω^{ab}: $\{z^a, z^b\} = \Omega^{ab}$.
Consider laplacian $\Delta_{1/2}$ on semidensities with principal symbol $S^{ab} = (-1)^a \Omega^{ab}$.

Laplacian on semidensities

Let M be an odd symplectic supermanifold equipped with non-degenerate Poisson bracket (anti-bracket) Ω^{ab}: $\{z^a, z^b\} = \Omega^{ab}$.
Consider laplacian $\Delta_{1/2}$ on semidensities with principal symbol $S^{ab} = (-1)^a \Omega^{ab}$.
In Darboux coordinates $z^A = \{x^i, \theta_j\}$ (x^i are even, θ_j are odd and $\{x^i, \theta_j\} = \delta^i_j$, $\{x^i, x^j\} = \{\theta_i, \theta_j\} = 0$. Laplacian on semidensities has the following appearance:

$$\Delta_{1/2} = \frac{1}{2}\left(S^{ab}\partial_b\partial_a + \partial_b S^{ba}\partial_a + \frac{1}{2}\partial_a\Gamma^a - \frac{1}{4}\Gamma^a\Gamma_a\right) =$$

$$\frac{\partial^2}{\partial x^i \partial \theta_i} + \frac{1}{4}\partial_a\Gamma^a - \frac{1}{8}\Gamma^a\Gamma_a = \frac{\partial^2}{\partial x^i \partial \theta_i} + U_\Gamma(x, \theta).$$

Changing of Laplacian under changing of connection

Difference of two connections is a vector field: $\Gamma' - \Gamma = \mathbf{X}$.
Consider cocycle $C_\Gamma(\mathbf{X}) = \Delta_{\Gamma'} - \Delta_\Gamma$ for $\Gamma' = \Gamma + \mathbf{X}$.

Changing of Laplacian under changing of connection

Difference of two connections is a vector field: $\Gamma' - \Gamma = \mathbf{X}$.
Consider cocycle $C_\Gamma(\mathbf{X}) = \Delta_{\Gamma'} - \Delta_\Gamma$ for $\Gamma' = \Gamma + \mathbf{X}$.

$$C_\Gamma(\mathbf{X}) = \Delta_{\Gamma'} - \Delta_\Gamma = (U_{\Gamma'} - U_\Gamma) = \frac{1}{4}\left(\partial_a \Gamma'^a - \partial_a \Gamma^a\right) -$$

$$\frac{1}{8}\left(\Gamma'_a \Gamma'^a - \Gamma_a \Gamma^a\right) = \frac{1}{4}\partial_a \mathbf{X}^a - \frac{1}{4}\Gamma_a X^a - \frac{1}{8}\mathbf{X}^2 =$$

$$\frac{1}{4}\left(\mathrm{div}_\Gamma \mathbf{X} - \frac{1}{2}\mathbf{X}^2\right)$$

(T.Voronov, H.Kh. 2003)

Useful anzats

Let **X** be a Hamiltonian vector field: $X_a = \frac{\partial F}{\partial x^a}$, ($\mathbf{X}^a = \Omega^{ab} X_b$).
Suppose for simplicity that $\Gamma = 0$ in given Darboux coordinates.
Then

$$C_\Gamma(\mathbf{X}) = \Delta_{\Gamma'} - \Delta_\Gamma = \frac{1}{4}\left(\operatorname{div}_\Gamma \mathbf{X} - \frac{1}{2}\mathbf{X}^2\right) =$$

$$\frac{1}{2}\frac{\partial^2}{\partial x^i \partial \theta_i} F - \frac{1}{4}\{F, F\} = -e^{F/2}\frac{\partial^2}{\partial x^i \partial \theta_i} e^{-F/2}$$

Useful anzats

Let **X** be a Hamiltonian vector field: $X_a = \frac{\partial F}{\partial x^a}$, $(\mathbf{X}^a = \Omega^{ab} X_b)$.
Suppose for simplicity that $\Gamma = 0$ in given Darboux coordinates.
Then

$$C_\Gamma(\mathbf{X}) = \Delta_{\Gamma'} - \Delta_\Gamma = \frac{1}{4}\left(\mathrm{div}_\Gamma \mathbf{X} - \frac{1}{2}\mathbf{X}^2\right) =$$

$$\frac{1}{2}\frac{\partial^2}{\partial x^i \partial \theta_i} F - \frac{1}{4}\{F, F\} = -e^{F/2}\frac{\partial^2}{\partial x^i \partial \theta_i} e^{-F/2}$$

Consider two examples

Action of canonical transformation on Laplacian

Let $f : z' = z'(z)$ be an arbitrary symplectomorphism of M (i.e. diffeomorphism which preserves Darboux coordinates): $z = (x, \theta) \to z' = (x', \theta')$. Principal symbol does not change.

Action of canonical transformation on Laplacian

Let $f: z' = z'(z)$ be an arbitrary symplectomorphism of M (i.e. diffeomorphism which preserves Darboux coordinates): $z = (x, \theta) \to z' = (x', \theta')$. Principal symbol does not change.

$$\text{Changing of connection:}\ \Gamma'_a = \Gamma^{(f)}_a = \frac{\partial z^{a'}}{\partial z^a}\left(\Gamma_a + \partial_{a'} \log \frac{\partial z}{\partial z'}\right).$$

Hence $X_a = -\partial_a \log J$ where $J = \mathrm{Ber}\,\frac{\partial(x',\theta')}{\partial(x,\theta)}$.
If $\Gamma = 0$ in Darboux coordinates (x, θ) then

$$c_\Gamma(\mathbf{X}) = \Delta_{\Gamma^f} - \Delta_\Gamma = \frac{1}{4}\left(\mathrm{div}_\Gamma \mathbf{X} - \frac{1}{2}\mathbf{X}^2\right) = -\frac{1}{\sqrt{J}}\frac{\partial^2}{\partial x^i \partial \theta_i}\sqrt{J}$$

Cocycle $c_\Gamma(\mathbf{X})$ vanishes due to the Batalin-Vilkovisky identity (1981):
$\frac{\partial^2}{\partial x^i \partial \theta_i}\sqrt{J} = \frac{\partial^2}{\partial x^i \partial \theta_i}\sqrt{\mathrm{Ber}\,\frac{\partial(x',\theta')}{\partial(x,\theta)}} = 0.$

Canonical operator on semidensities

We see that cocycle $c_\Gamma(\mathbf{X}) = \Delta_\Gamma^{(f)} - \Delta_\Gamma$ ($\mathbf{X} = \Gamma^{(f)} - \Gamma$) vanishes for an rbitrary symplectomorphism if $\Gamma = 0$ in some Darboux coordinates.

Thus we come to canonical operator on semidensities

$$\Delta = \frac{\partial^2}{\partial x^i \partial \theta_i}$$

(H.Kh. 1999)

Changing of connection induced by changing of volume form

Let $\Gamma = 0$ in given Darboux coordinates and let Γ' be a flat connection induced by arbitrary volume form $\rho(z)|Dz|$:

$$\Gamma'_a = -\partial_a \log \rho(z)\,.$$

Changing of connection induced by changing of volume form

Let $\Gamma = 0$ in given Darboux coordinates and let Γ' be a flat connection induced by arbitrary volume form $\rho(z)|Dz|$:

$$\Gamma'_a = -\partial_a \log \rho(z)\,.$$

Then using the anzats we have

$$C_\Gamma(\mathbf{X}) = \Delta_{\Gamma'} - \Delta_\Gamma = \frac{1}{4}\left(\mathrm{div}_\Gamma \mathbf{X} - \frac{1}{2}\mathbf{X}^2\right) = -\frac{1}{\sqrt{\rho}}\frac{\partial^2}{\partial x^i \partial \theta^i}\sqrt{\rho}$$

The cocycle $C_\Gamma(\mathbf{X})$ vanishes $\Leftrightarrow$ $\frac{\partial^2}{\partial x^i \partial \theta^i}\sqrt{\rho} = 0$, i.e. Batalin-Vilkovisky equation for $\rho = e^S$ is obeyed. $\Leftrightarrow$ There exists symplectomorphism f such that $\Gamma' = \Gamma^{(f)}$.

Comparison

R	Odd symplectic supermanifold.

Comparison

R

$$\Delta_\Gamma = \frac{\partial^2}{\partial x^2} - \frac{1}{2}\left(\Gamma' + \frac{1}{2}\Gamma^2\right)$$

Odd symplectic supermanifold.

$$\Delta_\Gamma = \frac{\partial^2}{\partial x^i \partial \theta_i} + \frac{1}{4}\left(\partial\Gamma - \frac{1}{2}\Gamma^2\right)$$

Comparison

R

$$\Delta_\Gamma = \frac{\partial^2}{\partial x^2} - \frac{1}{2}\left(\Gamma' + \frac{1}{2}\Gamma^2\right)$$

$$\mathscr{F}_{-1/2} \longrightarrow \mathscr{F}_{+3/2}$$

Odd symplectic supermanifold.

$$\Delta_\Gamma = \frac{\partial^2}{\partial x^i \partial \theta_i} + \frac{1}{4}\left(\partial\Gamma - \frac{1}{2}\Gamma^2\right)$$

$$\mathscr{F}_{1/2} \longrightarrow \mathscr{F}_{1/2}$$

Comparison

R

$$\Delta_\Gamma = \frac{\partial^2}{\partial x^2} - \frac{1}{2}\left(\Gamma' + \frac{1}{2}\Gamma^2\right)$$

$$\mathscr{F}_{-1/2} \to \mathscr{F}_{+3/2}$$

Odd symplectic supermanifold.

$$\Delta_\Gamma = \frac{\partial^2}{\partial x^i \partial \theta_i} + \frac{1}{4}\left(\partial\Gamma - \frac{1}{2}\Gamma^2\right)$$

$$\mathscr{F}_{1/2} \to \mathscr{F}_{1/2}$$

$$c_\Gamma(\mathbf{X}) = \Delta_{\Gamma'} - \Delta_\Gamma \text{ where } \Gamma' - \Gamma = \mathbf{X}$$

$$c_\Gamma(\mathbf{X}) = -\frac{1}{2}\mathrm{div}_\Gamma \mathbf{X} - \frac{1}{4}\mathbf{X}^2$$

$$c_\Gamma(\mathbf{X}) = \frac{1}{4}\mathrm{div}_\Gamma \mathbf{X} - \frac{1}{8}\mathbf{X}^2$$

Comparison

R

$$\Delta_\Gamma = \frac{\partial^2}{\partial x^2} - \frac{1}{2}\left(\Gamma' + \frac{1}{2}\Gamma^2\right)$$

$$\mathcal{F}_{-1/2} \to \mathcal{F}_{+3/2}$$

Odd symplectic supermanifold.

$$\Delta_\Gamma = \frac{\partial^2}{\partial x^i \partial \theta_i} + \frac{1}{4}\left(\partial\Gamma - \frac{1}{2}\Gamma^2\right)$$

$$\mathcal{F}_{1/2} \to \mathcal{F}_{1/2}$$

$c_\Gamma(\mathbf{X}) = \Delta_{\Gamma'} - \Delta_\Gamma$ where $\Gamma' - \Gamma = \mathbf{X}$

$$c_\Gamma(\mathbf{X}) = -\frac{1}{2}\mathrm{div}_\Gamma \mathbf{X} - \frac{1}{4}\mathbf{X}^2$$

$c_\Gamma(\mathbf{X})$ is a Schwarzian. It vanishes iff the new connection Γ' is such that $\Gamma' = \Gamma^{(f)}$ where f is a projective transformation

$$c_\Gamma(\mathbf{X}) = \frac{1}{4}\mathrm{div}_\Gamma \mathbf{X} - \frac{1}{8}\mathbf{X}^2$$

$c_\Gamma(\mathbf{X})$ is Batalin-Vilkovisky operator. It vanishes if the new connection Γ' is such that $\Gamma' = \Gamma^{(f)}$ where f is a symplectomorphism.

References

I.A.Batalin, G.A Vilkovisky. *Gauge algebra and quantization.* Phys.Lett., **102B** (1981), 27-31.

O.M. Khudaverdian, A.P. Nersessian. *On Geometry of Batalin-Vilkovisky Formalism.* Mod. Phys. Lett. A, v. 8 (1993), No. 25, p. 2377-2385.

A.S. Schwarz. *Geometry of Batalin Vilkovsiky formalism*. Comm.Math.Phys., **155**, (1993), no.2, 249–260.

H.M. Khudaverdian *Semidensities on odd symplectic supermanifold.*, Comm. Math. Phys., v. 247 (2004), pp. 353-390 (Preprint of Max-Planck-Institut für Mathematik, MPI-135 (1999), Bonn.)

H.M. Khudaverdian, T.Voronov *On odd Laplace operators. II.* In *Amer. Math. Soc. Transl.* (2), Vol. 212, 2004, pp.179–205