Geometrical foundations of the Batalin-Vilkovisky formalism.

Hovhannes Khudaverdian

University of Manchester, Manchester, UK

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Abstract...

The aim of this talk is to explain how the Batalin-Vilkvosiky (BV) formalism follows from the basic principles of field theory and geometry.
To obtain the partition function of the theory one needs to integrate the exponent of action functional over all fields. If a Lagrangian is degenerate (like for gauge theories), then by integrating exponent of the action first over symmetries one arrives to the integral of a non-local measure functional over the 'surface' defined in the space of fields by gauge conditions. In order to make this functional local one needs to expand the space of fields by ghosts. One comes finally to a gauge independent local action in the space of fields and ghosts. This is the famous 'Fadeev-Popov trick' which in particular works for Yang-Mills gauge theory.

...Abstract...

One can consider the surface of gauge conditions as a Lagrangian surface in the symplectic space of fields and anti-fields provided with the canoncial odd symplectic structure. In this case the measure functional over the surface of gauge conditions becomes half-density, the master-half-density, in this symplectic space. The gauge-independence can be formulated as a condition of vanishing of this master-half-density under the action of the canonical odd Laplacian. This is the complete describtion of the BV quantum master equation. The initial action and symmetries of the theory are boundary conditions which define this master half-density.

...Abstract

Such a formulation is equivalent to the Fadeev-Popov trick in the case of so called 'closed algebra of symmetries' (e.g. for Yang-Mills theory). On the other hand the formulation in terms of half-densities is invariant with respect to wider algebra of transformations, it works for an arbitrary degenerate Lagrangian, and it becomes necessary if we have so called 'open algebra of symmetries'. In the classical limit the quantum BV equation on master half-density becomes the well-known BV equation on the master action.
Finally we explain the Severa interpretation of the BV quantum master equation in terms of specially constructed spectral sequence.

Partition function in field theory

$$Z=\int e^{\frac{iS(\varphi)}{\hbar}}\mathscr{D}\varphi(x)$$

$$S(\varphi)=\int L(\varphi,\partial\varphi)d^4x\,.$$

$$Z=Z(j)=\int e^{\frac{i}{\hbar}\left(S(\varphi)+\int j(x)\varphi(x)d^4x\right)}\mathscr{D}\varphi(x)$$

$$G(x_1,x_2)=<\varphi(x_1)\varphi(x_2)>=\frac{\delta}{\delta j(x_1)}\frac{\delta}{\delta j(x_2)}\Big|_{j=0}Z(j)\,.$$

Finite-dimensional analog

$$\int e^{\frac{iS(\varphi)}{\hbar}} \mathscr{D}\varphi(x) \to \int e^{-F(\mathbf{x})} d^N x\,,$$

$$F(\mathbf{x}) = \langle \mathbf{x}, A\mathbf{x} \rangle + \text{terms of order} \geq 3 \text{ on } \mathbf{x} =$$

$$A_{mn} x^m x^n + \sum_{k \geq 3} c_{i_1 \ldots i_k} x^{i_1} \ldots x^{i_k}\,.$$

$$\int e^{-F(\mathbf{x})} d^N x = \sum_k \int e^{-\langle \mathbf{x}, A\mathbf{x} \rangle} \tilde{c}_{i_1 \ldots i_k} x^{i_1} \ldots x^{i_k} d^N x$$

$$\int e^{-\langle \mathbf{x}, A\mathbf{x} \rangle} x^{i_1} \ldots x^{i_k} d^N x = \frac{\partial}{\partial j_{i_1}} \ldots \frac{\partial}{\partial j_{i_k}} \Big|_{j=0} \int e^{-\langle \mathbf{x}, A\mathbf{x} \rangle + j\mathbf{x}} d^N x\,.$$

Calculation of integral $\int e^{-\langle \mathbf{x}, A\mathbf{x}\rangle + j_k x^k} d^N x$

$$\int e^{-\langle \mathbf{x}, A\mathbf{x}\rangle + j\mathbf{x}} d^N x = e^{-\langle \mathbf{l}, A\mathbf{l}\rangle} \int e^{-\langle \mathbf{x}, A\mathbf{x}\rangle + (j - 2\mathbf{l}A)\mathbf{x}} d^N x =$$

$$\mathbf{x} \to \mathbf{x} + \mathbf{l}$$

Take $\mathbf{l} = \frac{1}{2} A^{-1} j$. Then $(j - 2\mathbf{l}A)\mathbf{x} \equiv 0$ and

$$\int e^{-\langle \mathbf{x}, A\mathbf{x}\rangle + j\mathbf{x}} d^N x = e^{-\frac{1}{4}\langle j, A^{-1} j\rangle} \int e^{-\langle \mathbf{x}, A\mathbf{x}\rangle} d^N x = C e^{-\frac{1}{4}\langle j, A^{-1} j\rangle},$$

$$\left(C = \int e^{-\langle \mathbf{x}, A\mathbf{x}\rangle} d^N x = \sqrt{\frac{\pi^N}{\det A}} \right).$$

This works in the case if operator A is non-degenerate.

We have performed calculations considering expansion of $F(\mathbf{x})$

$$F(\mathbf{x}) = \langle \mathbf{x}, A\mathbf{x} \rangle + \text{terms of order} \geq 3 \text{ on } \mathbf{x} =$$

in a vicinity of stationary point in the case if $HessianF = A$ is non-degenerate

$$\downarrow$$

For functional integral $\int e^{\frac{i}{\hbar}S(\varphi)} \mathscr{D}\varphi$ one has to consider quadratic expansion in a vicinity of stationary points $\frac{\delta S}{\delta \varphi} = 0$ (classical equations of motion) and this expansion has to be non-degenerate.

What happens if this is not the case?

Language of condensed notations

$$S(\varphi(x)) \to S(\varphi^i),$$

$$\text{equations of motion } \mathscr{F}(x) = \frac{\delta S(\varphi)}{\delta \varphi(x)} \to \frac{\partial S(\varphi)}{\partial \varphi^i}.$$

We use the language of condensed notations. Index 'i' runs over all discrete and continuous indices. E.g. in this language a function $\varphi(x)$ is the collection of $\{\varphi^i\}$, variational derivative $\frac{\delta}{\delta\varphi(x)}$ becomes 'partial derivative' $\frac{\partial}{\partial\varphi^i}$

Degenerate Lagrangian—Gauge Theory

$$S = S(\varphi^i) \qquad \mathscr{F}_i = \frac{\partial S(\varphi)}{\partial \varphi^i} = 0 \quad \text{class.equations of motion}\,.$$

$$M_{\text{station.}} = \{\varphi^i : \quad \mathscr{F}_i(\varphi) = 0\}$$

The action $S(\varphi)$ is degenerate if $\frac{\partial \mathscr{F}_i(\varphi)}{\partial \varphi^j} = \frac{\partial^2 S(\varphi)}{\partial \varphi^i \partial \varphi^j}\Big|_{M_{\text{station.}}}$ is degenerate.

$$\operatorname{rank} \frac{\partial \mathscr{F}_i(\varphi)}{\partial \varphi^j} + \dim M_{\text{stat.}} = \text{‘number of fields’}\,.$$

Local (gauge) symmetries

'dimension of $M_{\rm station.}$='number of gauge symmetries'

Symmetries: Set of vector fields $\{\mathbf{R}_\alpha = R^i_\alpha \frac{\partial}{\partial \varphi^i}\}$:

$$\mathbf{R}_\alpha S(\varphi) = R^i_\alpha \frac{\partial S(\varphi)}{\partial \varphi^i} = R^i_\alpha \mathscr{F}_i = 0\,, \quad S(\varphi^i + \varepsilon^\alpha R^i_\alpha) = S(\varphi^i)\,,$$

$$R^i_\alpha\big|_{M_{\rm station.}} \neq 0\,,$$

These are local (gauge) symmetries (Noether identities).
The index α ('number' of symmetries runs over continuous set.)
The global symmetries are excluded out of considerations

...Local symmetries

$$0 \longrightarrow F \longrightarrow E \longrightarrow B \longrightarrow 0$$

E—symmetries, vector fields **R** preserving action:

$$\mathbf{R}(\varphi)S(\varphi) = R^i(\varphi)\frac{\partial S(\varphi)}{\partial \varphi^i} = R^i \mathscr{F}_i = 0\,.$$

F—space of symmetries which vanish on-shell:

$$\mathbf{R} \in F \,\mathrm{if}\, \mathbf{R}\big|_{M_{\mathrm{station.}}} = 0 \leftrightarrow \mathbf{R} = R^i(\varphi)\frac{\partial}{\partial \varphi^i} = E^{ij}(\varphi)\mathscr{F}_j(\varphi)\frac{\partial}{\partial \varphi^i}\,,$$

$$E^{ij}(\varphi) = -E^{ji}(\varphi)\,.$$

Open and closed algebras of symmetries

$\{\mathbf{R}_\alpha\}$—symmetries. Commutator is also a symmetry:

$$[\mathbf{R}_\alpha, \mathbf{R}_\beta] = t^\gamma_{\alpha\beta}(\varphi)\mathbf{R}_\gamma + E^{ij}_{\alpha\beta}(\varphi)\mathscr{F}_j(\varphi)\frac{\partial}{\partial\varphi^j}\,.$$

$E^{ij}_{\alpha\beta} \not\equiv 0$— open algebra of symmetries ('on-shell' symmetries)
$E^{ij}_{\alpha\beta} \equiv 0$— closed algebra of symmetries ('of-shell' symmetries)

$$\mathbf{R}_\alpha \to \lambda^\beta_\alpha(\varphi)\mathbf{R}_\beta + F^{[ij]}_\alpha(\varphi)\mathscr{F}_j$$

Abelization of symmetries

Consider transformation

$$\mathbf{R}_\alpha \to \widetilde{\mathbf{R}}_\alpha = \lambda_\alpha^\beta(\varphi)\mathbf{R}_\beta + F_\alpha^{[ij]}(\varphi)\mathscr{F}_j$$

such that

$$\left[\widetilde{\mathbf{R}}_\alpha, \widetilde{\mathbf{R}}_\beta\right] \equiv 0 \,.$$

We simplify the theory making symmetries abelian.
On the other hand we pay the enormous price making these symmetries non-local

It will be great to have a formalism which 'allows' these transformations.

This is the Batalin-Vilkovisky (BV) formalism.

Return to functional integral

$Z = \int e^{\frac{iS(\varphi)}{\hbar}} \mathscr{D}\varphi(x)$, $(S(\varphi) = \int L(\varphi, \partial\varphi) d^4 x$.
In condensed notation $Z = \int e^{\frac{iS(\varphi)}{\hbar}} \mathscr{D}\varphi$, $(\mathscr{D}\varphi = \wedge_i d\varphi^i)$.
For degenerate Lagrangian: $M_{\text{station.}} = \{\varphi^i : \mathscr{F}_i(\varphi) = 0\}$,

$$\mathbf{R}_\alpha S(\varphi) = R^i_\alpha \frac{\partial S(\varphi)}{\partial \varphi^i} = R^i_\alpha \mathscr{F}_i = 0\,, \quad \left(\mathscr{F}_i = \frac{\partial S(\varphi)}{\partial \varphi^i}\right).$$

'dimension' of $M_{\text{station.}}$ = 'dimension' of algebra of symmetries

Surface of gauge conditions

In the space of fields consider a surface $C_{\rm gauge}$ transversal to symmetries $\{\mathbf{R}_\alpha\}$,

$$C_{\rm gauge}: \quad \Psi_\alpha = 0\,.$$

Action $S(\varphi)$ is preserved along symmetries. Integrate $e^{\frac{i}{\hbar}S(\varphi)}$ along symmetries. We come to non-degenerate measure on the surface C_g of gauge conditions.
How does this measure look?

à la Fadeev-Popov trick

C_{gauge} —gauge surface. $\int e^{\frac{i}{\hbar}S(\varphi)}\mathscr{D}\varphi =$

$$\int \underbrace{\mathscr{D}\varphi}_{\text{integral over } C_{\text{gauge}}} \underbrace{\left[\int e^{\frac{i}{\hbar}S(\varphi^g)}\mathscr{D}g\right]}_{\text{integral over symmetries}} = \int \underbrace{\mu(\varphi)\mathscr{D}\varphi}_{\text{integral over } C_{\text{gauge}}}.$$

This is gauge independent (but it is non-local).

Toy example

Let **K** be a compact[1] vector field on a space E:

$$E = \mathbf{R}^{N-1} \times S^1\,, \quad \mathbf{K} = \frac{\partial}{\partial \varphi}\,,$$

$x^1, x^2, \dots, x^{N-1}, x^N$—coordinates on E, $x^N = \varphi$
Let $\underline{\rho} = \rho(x) dx^1 \wedge \dots \wedge dx^N$ be a volume form E.
Suppose that $\underline{\rho}$ is invariant with respect to **K**, $\mathscr{L}_{\mathbf{K}} \underline{\rho} = 0$.
Let $C = C_{\mathrm{gauge}}$ be $N-1$-dimensional surface in E transversal to **K**, defined by equation $\Psi(\mathbf{x}) = 0$. Then

$$\int_E \underline{\rho} = \underbrace{2\pi}_{\text{volume of } U(1)} \int_{C_{\mathrm{gauge}}} \imath_{\mathbf{K}} \underline{\rho} = 2\pi \int_E K^i \frac{\partial \Psi(x)}{\partial x^i} \delta(\partial \Psi(x)) \rho(x) d^N x\,.$$

[1] it generates action of group $U(1)$

Gauss-Ostrogradsky Law

$$Z_\Psi = \int_E K^i \frac{\partial \Psi(x)}{\partial x^i} \delta(\Psi(x)) \rho(x) d^N x = \int_{C_{\text{gauge}}} \mathbf{K}\underline{\rho}\,,$$

is a flux of vector density $\mathbf{K}\underline{\rho}$ trough the surface C.
$\mathscr{L}_\mathbf{K}\underline{\rho} = d \circ \iota_\mathbf{K}\underline{\rho} = 0 \leftrightarrow \operatorname{div}_{\underline{\rho}} \mathbf{K} = 0$.
Z_Ψ does not depend on gauge conditions: $Z_{\Psi+\delta\Psi} = Z_\Psi$.

$$\text{Put } \underline{\rho} = \rho(x) dx^1 \wedge \cdots \wedge dx^N = e^{\frac{i}{\hbar} S(x)} dx^1 \wedge \cdots \wedge dx^N\,,$$

$$0 = \operatorname{div}_{\underline{\rho}} \mathbf{K} = \frac{\partial K^i}{\partial x^i} + \frac{i}{\hbar} K^i \frac{\partial S(x)}{\partial x^i}\,,$$

$\mathscr{L}_\mathbf{K} S = K^i \partial_i S(x) = 0$. This is Noether identity for ‘action’ $S(x)$.

Flux of multivector field-density

Volume form $\underline{\rho} = e^{\frac{i}{\hbar}S(x)} dx^1 \wedge \cdots \wedge dx^N$ is invariant with respect to vector fields $\{\mathbf{K}_a\}$ $(a = 1, \ldots, k)$.
C_{gauge}—$N-k$-dimensional surface transversal to vector fields

$$C_{gauge}: \ \Psi^b = 0 (b = 1, \ldots, k), \, Z = \int_E \underline{\rho} = (2\pi)^k \int_{C_{\rm gauge}} \mathbf{K}_1 \wedge \cdots \wedge \mathbf{K}_k \underline{\rho}$$

$$= \int \det \left(\mathbf{K}_a^i \frac{\partial \Psi^b(x)}{\partial x^i} \right) \prod_a \delta(\Psi^a) e^{\frac{i}{\hbar}S(x)} d^N x$$

$Z =$ flux of multivector density through gauge surface
This density is 'prepared' from gauge symmetries
Gauge independence $=$ divergenceless of the density

Return to functional integral

$$Z = \int \det\left(\mathbf{R}^i_\alpha \frac{\partial \Psi^\beta(\varphi)}{\partial \varphi^i}\right) \prod_\gamma \delta(\Psi^\gamma(\varphi)) e^{\frac{i}{\hbar}S(\varphi)} \mathscr{D}\varphi$$

Partition function is the integral of multivector density

$$\left(\underbrace{\wedge_\alpha \mathbf{R}_\alpha}_{\textit{multivector field}} \otimes \underbrace{\underline{\rho}}_{\textit{density}}\right)$$ over gauge fixing surface $C_{\text{gauge}}: \ \Psi^\alpha = 0$,

$$\underline{\rho} = e^{\frac{i}{\hbar}S(\varphi)} \mathscr{D}\varphi\,.$$

This is gauge invariant expression, but functional is non-local, indices α, β, γ run over continuous set.

Localising of functional integral

$$\det A = \int e^{A_{ik}\theta^i\bar{\theta}^k} d^n\theta d^n\bar{\theta}\,, \quad (A\text{–}n \times n \text{ matrix})\,,$$

and $\delta(x) = \frac{1}{2\pi}\int \mathbf{e}^{ikx} dk$. Hence

$$Z = \int \det\left(\mathbf{R}^i_\alpha \frac{\delta\Psi^\beta(\varphi)}{\delta\varphi^i}\right) \prod_\gamma \delta(\Psi^\gamma(\varphi)) e^{\frac{i}{\hbar}S(\varphi} \mathscr{D}\varphi =$$

$$= \int e^{\frac{i}{\hbar}S(\varphi) + c^\alpha \mathbf{R}^i_\alpha \frac{\partial\Psi^\beta}{\delta\varphi^i}\eta_\beta + \lambda_\alpha\Psi^\alpha} \mathscr{D}\varphi \mathscr{D}c \mathscr{D}\eta \mathscr{D}\lambda$$

Space of fields and antifields

The aim is to construct multivector density in the general case when symmetries are not abelian. We raise integral:

$$Z = \int_{\Psi^a=0} \textit{multivector density}$$

to the space of fields and anti-fields.
Consider ΠT^*M (space of fields and antifields), (M is space of fields φ^i). Denote by φ_* coordinates of fibre: $p(\varphi_{*i}) = p(\varphi^i) + 1$. $(\varphi^i, \varphi_{*j})$ are Darboux coordinates of odd symplectic superspace ΠT^*M. Corresponding canonical odd Poisson bracket:

$$\{\varphi^i, \varphi_{*j}\} = \delta^i_j\,, \quad \{\varphi^i, \varphi^j\} = \{\varphi_{*i}, \varphi_{*j}\} = 0\,.$$

Surface of gauge conditions → Lagrangian surface in $\Pi T^* M$

To surface $C_{\rm gauge} = \{\varphi\colon \Psi^\alpha(\varphi) = 0\}$ in the space M of fields we assign a Lagrangian surface $L_{C_{\rm gauge}}$ in the space $\Pi T^* M$ of fields and anti-fields

$$L_C = \left\{ \varphi_{*i} = \frac{\partial \Psi(\varphi,\eta)}{\partial \varphi^i} = \frac{\partial \Psi^\alpha(\varphi)}{\partial \varphi^i}\eta_\alpha, \frac{\partial \Psi(\varphi,\eta)}{\partial \eta_a} = \Psi^\alpha(\varphi) = 0 \right\},$$

where $\Psi(\varphi) = \Psi^\alpha(\varphi)\eta_\alpha$, ($\eta_\alpha$ are odd parameters).

$(N-k)$-dimensional surface C defines
$(N-k, N-k)$-dimensional Lagrangian surface L_C.

$\Psi(\varphi)$ is called gauge fermion

Multivector density $\to$ Half-density

Density $\underline{\rho}$ on M is an half density on symplectic space ΠT^*M
Multivector field $\wedge_\alpha \mathbf{R}$ on M is a function on ΠT^*M
Hence multivector density on M is an half-density on ΠT^*M

Example

Let $(\varphi^i, \varphi_{*j})$ be coordinates of ΠT^*E. Vector field $\mathbf{K} = K^i \frac{\partial}{\partial \varphi^i}$ on E is a function $K^i(\varphi)\varphi_{*i}$ on ΠT^*E. Vector density $\mathbf{K}\underline{\rho}$ is a half-density in odd symplectic superspace ΠT^*E:

$$s_{\mathbf{K}} = K^i(\varphi)\varphi_{*i}\rho(\varphi)\sqrt{D(\varphi,\varphi_*)}$$

If $\begin{cases} \varphi^{i'} = \varphi^{i'}(\varphi^i) \\ \varphi_{*i'} = \frac{\partial \varphi^i}{\partial \varphi^{i'}} \varphi_{*i} \end{cases}$ then $\mathrm{Ber}\left(\frac{\partial(\varphi,\varphi_*)}{\partial(\varphi',\varphi'_*)}\right) = \begin{pmatrix} \frac{\partial \varphi^{i'}}{\partial \varphi^i} & \frac{\partial \varphi_{*j'}}{\partial \varphi^j} \\ 0 & \frac{\partial \varphi_{*j'}}{\partial \varphi_{*j}} \end{pmatrix} = \left(\det\left(\frac{\partial \varphi^{i'}}{\partial \varphi^i}\right)\right)^2 .$

Partition function—integral of half-density over Lagrangian surface

$$Z = \int \det\left(\mathbf{R}^i_\alpha \frac{\partial \Psi^\beta(\varphi)}{\partial \varphi^i}\right) \prod_\gamma \delta(\Psi^\gamma(\varphi)) e^{\frac{i}{\hbar}S(\varphi)} \mathscr{D}\varphi$$

$$\int_{\text{surface } C:\ \Psi^\alpha=0} \underbrace{\wedge \mathbf{R}_\alpha \underline{\rho}}_{\text{multivector density}} = \int_{\text{Lagrangian surface } L_C} \underbrace{\mathbf{s}}_{\text{half-density}}$$

$$\underline{\rho} = e^{\frac{i}{\hbar}S(\varphi)} \mathscr{D}(\varphi), \qquad \mathbf{s} = e^{\frac{i}{\hbar}\mathscr{S}(\varphi,\varphi_*)} \sqrt{\mathscr{D}(\varphi,\varphi_*)},$$

$$\mathscr{S}(\varphi,\varphi_*) = S(\varphi) + c^\alpha R^i_\alpha(\varphi)\varphi_{*i} + \dots$$

Gauge independence
$\mathrm{div}_{\underline{\rho}} \wedge_\alpha \mathbf{R} = 0$

How is it expressed in terms of half-densities?

Theorem

(H.Kh.(1999)) In an odd symplectic supermanifold there exists canonical odd operator Δ *acting on half-densities. In local Darboux coordinates* $(\varphi^i, \varphi_{*j})$

$$\Delta\left(s(\varphi,\varphi_*)\sqrt{\mathscr{D}(\varphi,\varphi_*)}\right)=\sum_i \frac{\partial^2 s(\varphi,\varphi_*)}{\partial\varphi^i\varphi_{*i}}\sqrt{\mathscr{D}(\varphi,\varphi_*)}\,.$$

Consider $\mathbf{s}=1\cdot\sqrt{\mathscr{D}(\varphi,\varphi_*)}$. Obviously $\Delta\mathbf{s}=0$. We come to

Corollary

(Batalin-Vilkovisky identity (1981)) Let $(\varphi,\varphi_*)\to(\varphi',\varphi'_*)$ *be an arbitrary canonical transformation, then*

$$\sum\frac{\partial^2}{\partial\varphi^i\partial\varphi_{*i}}\sqrt{\mathrm{Ber}\left(\frac{\partial\left(\varphi',\varphi'_*\right)}{\partial\left(\varphi,\varphi_*\right)}\right)}=0\,.$$

Relation of operator Δ on half-densities with operator $\Delta_{\mathscr{D}\mathbf{v}}$ on functions

$$\Delta_{\mathscr{D}\mathbf{v}} F = \frac{1}{2} \mathrm{div}_{\mathscr{D}\mathbf{v}} \widehat{D_F},$$

where $\mathscr{D}\mathbf{v}$ is a volume form on odd sympelctic supermanifold, and $\widehat{D}_F$ is Hamiltonian vector field defined by F, $\widehat{D}_F G = \{F, G\}$. In Darboux coordinates $\{\varphi^i, \varphi_{*j}\}$

$$\Delta_{\mathscr{D}\mathbf{v}} F = \frac{\partial^2 F(\varphi, \varphi_*)}{\partial \varphi^i \partial \varphi_{*i}} + \frac{1}{2}\{\log V, F\},$$

where $\mathscr{D}\mathbf{v} = V(\varphi, \varphi_*) \mathscr{D}(\varphi, \varphi_*)$

$$\Delta(F\mathbf{s}) = F\Delta(\mathbf{s}) + (-1)^{p(F)} (\Delta_{\mathscr{D}\mathbf{v}} F)$$

If $\mathbf{s}$ is half-density such that $\mathbf{s}^2 = \mathscr{D}\mathbf{v}$ is a volume form then

Nilpotency of operators Δ and $\Delta_{\mathscr{D}\mathbf{v}}$. Relation with $\{\,,\,\}$

$$\Delta^2 = 0\,,$$

$$\Delta^2_{\mathscr{D}\mathbf{v}}(F) = \left\{ \frac{1}{\sqrt{\mathscr{D}\mathbf{v}}} \Delta\left(\sqrt{\mathscr{D}\mathbf{v}}\right), F \right\},$$

$$\Delta_{\mathscr{D}\mathbf{v}}\{F, G\} = \{\Delta_{\mathscr{D}\mathbf{v}} F, G\} + (-1)^{p(f)+1}\{F, \Delta_{\mathscr{D}\mathbf{v}} G\}$$

$$\Delta_{\mathscr{D}\mathbf{v}}(F \cdot G) = \Delta_{\mathscr{D}\mathbf{v}} F \cdot G + (-1)^{p(f)} F \cdot \Delta_{\mathscr{D}\mathbf{v}} G + (-1)^{p(f)}\{F, G\}\,.$$

Divergence $\to$ Δ-operator

In a special case if $\underline{\rho} = \rho(\varphi)$ is a volume form on M and a semidensity $\mathbf{s}_{\underline{\rho}} = \rho(\varphi)\sqrt{\mathscr{D}(\varphi, \varphi_*)}$ then for a multivector field $\mathbf{P} = P^{a_1 \dots a_k} \partial_{a_1} \wedge \dots \wedge \partial_{a_k} = P^{a_1 \dots a_k} \varphi_{* a_1} \dots \varphi_{* a_k}$,

$$\operatorname{div}_{\underline{\rho}} K\mathbf{s} = \Delta\left(\mathbf{P}\mathbf{s}_{\underline{\rho}}\right)$$

multivector density on $\Pi T^* M \leftrightarrow$ differential form on M, $\Delta \leftrightarrow$ de Rham differential.

If half-density $\mathbf{s}$ corresponds to multivector density $\mathbf{K}$ then

$$\operatorname{div} \mathbf{K} = 0 \to \Delta\mathbf{s} = \mathbf{0}\,.$$

$$(\mathbf{K} = \mathbf{P} \otimes \underline{\rho}, \mathbf{s} = \mathbf{P} \otimes \mathbf{s}_{\underline{\rho}})$$

Master-equation for Partition function

Returning to partiion function

$$Z = \int \det\left(\mathbf{R}^i_\alpha \frac{\partial \Psi^\beta(\varphi)}{\partial \varphi^i}\right) \prod_\gamma \delta(\Psi^\gamma(\varphi)) e^{\frac{i}{\hbar}S(\varphi)} \mathscr{D}\varphi$$

$$\int_{\text{surface } C:\ \Psi^\alpha = 0} \underbrace{\wedge \mathbf{R}_\alpha \underline{\rho}}_{\text{multivector density}} = \int_{\text{Lagrangian surface } L_C} \underbrace{\mathbf{s}}_{\text{half-density}}$$

$$\underline{\rho} = e^{\frac{i}{\hbar}S(\varphi)} \mathscr{D}(\varphi), \qquad \mathbf{s} = e^{\frac{i}{\hbar}\mathscr{S}(\varphi,\varphi_*)} \sqrt{\mathscr{D}(\varphi,\varphi_*)},$$

$$\mathscr{S}(\varphi,\varphi_*) = S(\varphi) + c^\alpha R^i_\alpha(\varphi)\varphi_{*i} + \dots$$

Gauge independence — How is it expressed?

$$\operatorname{div}_{\underline{\rho}} \wedge_\alpha \mathbf{R} = 0 \qquad \Delta \mathbf{s} = \mathbf{0}$$

BV-master-equation

$$\Delta \mathbf{s} = \mathbf{0}\,,$$

where

$$\mathbf{s} = \mathbf{e}^{\frac{\mathbf{i}}{\hbar}\mathscr{S}(\varphi,\varphi_*)}\sqrt{\mathscr{D}(\varphi,\varphi_*)}\,,$$

with boundary conditions:

$$\mathscr{S}(\varphi,\varphi_*) = S(\varphi) + c^\alpha R^i_\alpha(\varphi)\varphi_{*i}i + \dots\,,$$

where $\mathbf{R}_\alpha = R^i_\alpha \frac{\partial}{\partial \varphi^i}$ are initial symmetries.

Quasiclassical approximation

$$\text{Master equation is } \Delta\left(e^{\frac{i}{\hbar}\mathscr{S}(\varphi,\varphi_*)}\sqrt{\mathscr{D}(\varphi,\varphi_*)}\right) =$$

$$\left(\frac{i}{\hbar}\frac{\partial^2\mathscr{S}}{\partial\varphi^i\varphi_{*i}} - \frac{1}{2\hbar^2}\{\mathscr{S},\mathscr{S}\}\right)e^{\frac{i}{\hbar}\mathscr{S}(\varphi,\varphi_*)}\sqrt{\mathscr{D}(\varphi,\varphi_*)} = 0\,.$$

If $\hbar \to 0$ then we come to classical BV-equation:

$$\{\mathscr{S},\mathscr{S}\} = 0\,.$$

Solution of classical BV-equation defines odd Hamiltonian vector field $\hat{Q}$: $\hat{Q}F = \mathscr{S}$ which defines the structure of Q-manifold on the space of fields and anti-fields.

Invariance of BV-master equation

Let $S(\varphi)$ be an action

Let $\mathbf{R}_\alpha$ be an abelian algebra of symmetries

$$[\mathbf{R}_\alpha, \mathbf{R}_\beta] = 0, \text{then}$$

$$C_{\text{gauge}}: \quad \Psi^a = 0$$

$$Z = \int_{C_{\text{gauge}}} \mathbf{P}$$

$\mathbf{P}$ is a multivector density

$$\mathbf{P} = \wedge_\alpha \mathbf{R}_\alpha e^{\frac{i}{\hbar} S(\varphi)} \mathscr{D}\varphi$$

$$\operatorname{div} \mathbf{P} = 0$$

Go to an open algebra of symmetries

$$\mathbf{R}_\alpha \to \lambda_\alpha^\beta(\varphi) \mathbf{R}_\beta + F_\alpha^{[ij]}(\varphi) \mathscr{F}_j$$

$$[\mathbf{R}_\alpha, \mathbf{R}_\beta] = t_{\alpha\beta}^\gamma(\varphi) \mathbf{R}_\gamma + E_{\alpha\beta}^{ij}(\varphi) \mathscr{F}_j(\varphi) \partial_j \varphi^j$$

$$\mathbf{P} \to ?$$

$$\mathscr{S}(\varphi, \varphi_*) = S(\varphi)$$

$$= + c_\alpha R_\alpha^i(\varphi) \varphi_{*i}$$

$$L_C: \; \varphi_{*i} = \partial_i \Psi^a \eta_\alpha$$

$$Z = \int_{L_C} \mathbf{s}$$

$\mathbf{s}$ is a half-density

$$\mathbf{s} = e^{\frac{i}{\hbar} \mathscr{S}(\varphi, \varphi_*)} \sqrt{\mathscr{D}(\varphi, \varphi_*)}$$

$$\Delta \mathbf{s} = 0$$

canonic. transformat.

master-equation remains

$$\mathscr{S} \to \mathscr{S})$$

$$= S(\varphi) + c^\alpha R_\alpha^i \varphi_{*i} + \dots$$

Finally two words about how to describe Δ operator on half-densities in invariant way?

(The original formula $\Delta \mathbf{s} = \frac{\partial^2 s(\varphi,\varphi_*)}{\partial \varphi^i \partial \varphi_{*i}} \sqrt{\mathscr{D}\varphi, \varphi_*}$ is written in Darboux coordinates).

Δ-operator and Severa's spectral sequence

Let $\Omega(M)$ be a space of all (pseudo)differential forms on ΠT^*M, i.e. functions $F(\varphi,\varphi_*,d\varphi,d\varphi_*)$ ($d\varphi, d\varphi_*$ have parity reverse to parity of φ,φ_*, φ and $d\varphi_*$ are even $d\varphi$ and φ_* are odd

Consider differential $Q=d+\omega$, where d is de Rham differential and $\omega = d\varphi^i d\varphi_{*i}$ defines canonical symplectic structure on ΠT^*M:

$$QF=(d+\omega)F(\varphi,\varphi_*,d\varphi,d\varphi_*)=$$

$$\underbrace{\left(d\varphi^i\frac{\partial F(\varphi,\varphi_*)}{\partial\varphi^i}+d\varphi_{*i}\frac{\partial F(\varphi,\varphi_*)}{\partial\varphi_{*i}}\right)}_{\text{dF}}+\underbrace{\left(d\varphi^i d\varphi_{*_i}F(\varphi,\varphi_*)\right)}_{\omega F},$$

$$Q^2=d^2=\omega^2=0, d\omega+\omega d=0\,.$$

Consider spectral sequence $\{E_r, d_r\}$

$$E_{r+1} = H(E_r, d_r)$$

with $E_0 = \Omega(M)$, $d_0 = \omega$.
The space $E_1 = H(\Omega(M), \omega)$ can be naturally identified with the space of semidensities on M:

$$\mathbf{s} = [s(\varphi, \varphi_*) d\varphi^1 d\varphi^2 \dots d\varphi^N]$$

$(\omega\mathbf{s} = \mathbf{0}, \mathbf{s} \neq \omega\mathbf{f})$

$$\mathbf{s} = \qquad s(\varphi, \varphi_*) d\varphi^1 d\varphi^2 \dots d\varphi^N$$

$$\downarrow d$$

$$(-1)^k \frac{\partial s}{\partial \varphi_{*k}} d\varphi^1 \dots d\varphi^{k-1} d\varphi^{k+1} \dots d\varphi^N \xrightarrow{\omega} \frac{\partial s}{\partial \varphi_{*i}} d\varphi_{*i} d\varphi^1 d\varphi^2 \dots d\varphi^N$$

$$\downarrow d$$

$$\Delta \mathbf{s} = \frac{\partial^2 s}{\partial \varphi^i \partial \varphi_{*l}} d\varphi^1 d\varphi^2 \dots d\varphi^N$$

The differential d_1 of the Severa's spectral sequence vanishes and differential d_2 coincides with the canonical operator Δ.

Thank you

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