PFAFFIANS IN ODD SYMPLECTIC GEOMETRY

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SUPERSYMMETRIES AND QUANTUM SYMMETRIES
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Contents

What is Pfaffian of matrix

Let K be an antisymmetrical matrix:

$$K^{+} = -K\,.$$

Then

$$\det K = (\mathrm{Pf}\,(K))^{2}\,,\ \sqrt{\det K} = \mathrm{Pf}\,(K)\,,$$

where $\mathrm{Pf}\,(K)$, Pfaffian of matrix K is a polynomial of entries of matrix K

Examples

If m is an odd number then $\mathrm{Pf}(K) = 0$, since $\det K = 0$:

$$\det K^{+} = \det K = (-1)^m \det K = -\det K\,.$$

$m = 2$

$$K = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix},$$

$$\det K = a^2, \mathrm{Pf}(K) = \sqrt{\det K} = a$$

Examples ($m = 4$)

$$K = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix}, \det K = (af + cd - be)^2$$

$$\mathrm{Pf}(K) = af + cd - be = K_{12}K_{34} + K_{14}K_{23} - K_{13}K_{24} \, .$$

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$$F = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & H_z & -H_y \\ -E_y & -H_z & 0 & H_x \\ -E_z & H_y & -H_x & 0 \end{pmatrix}$$

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$$F \wedge F = \mathrm{Pf}(F) dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$$

Odd canonical transformations

$n|n$-dimensional odd symplectic superspace:
$\{x^1, \ldots, x^n; \theta_1, \ldots, \theta_n\}$

$$\omega = dx^a d\theta_a \qquad (*)$$

$$\{f, g\} = \frac{\partial f}{\partial x^a}\frac{\partial g}{\partial \theta_a} + (-1)^{p(f)}\frac{\partial f}{\partial \theta_a}\frac{\partial g}{\partial x^a} \qquad (**)$$

$$\{x^a, \theta_b\} = \delta^a_b \, \{x^a, x^b\} = 0, , \{\theta_a, \theta_b\} = 0,$$

$\{x^1, \ldots, x^n; \theta_1, \ldots, \theta_n\}$ are Darboux coordinates

Odd canonical transformation preserve the form (*)
(the odd Poisson bracket (**))

Linear odd canonical transformation

$$(x,\theta)\to(y,\eta)=(x,\theta)\begin{pmatrix} A & \mathscr{B} \\ \mathscr{C} & D \end{pmatrix},\begin{cases} y^a = x^b A^a_b + \theta_b \mathscr{C}^b_a \\ \eta_a = x^b \mathscr{B}_{ba} + \theta_b D^b_a \end{cases}$$

where entries of $n\times n$ matrices A and D are even numbers (even elements of a Grassmann algebra), and entries of $n\times n$ matrices $\mathscr{B}$ and $\mathscr{C}$ are odd numbers (odd elements of a Grassmann algebra) and the following conditions are obeyed:

$$\begin{cases} A^+\mathscr{C}+\mathscr{C}^+A=0 \\ D^+\mathscr{B}=\mathscr{B}^+D \\ A^+D+\mathscr{C}^+\mathscr{B}=1 \end{cases}$$

Examples

$$K = \begin{pmatrix} A & \mathscr{B} \\ \mathscr{C} & D \end{pmatrix}: \qquad \begin{cases} A^{+}\mathscr{C} + \mathscr{C}^{+}A = 0 \\ D^{+}\mathscr{B} = \mathscr{B}^{+}D \\ A^{+}D + \mathscr{C}^{+}\mathscr{B} = 1 \end{cases}$$

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$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \qquad A^{+}D=1$$

$$\begin{pmatrix} 1+\mathscr{B}\mathscr{C} & \mathscr{B} \\ \mathscr{C} & 1 \end{pmatrix} \qquad \mathscr{B}^{+}=\mathscr{B}, \mathscr{C}^{+}=-\mathscr{C}\,.$$

Berezinian of odd canon.transform

In a drastic difference to the even case odd canonical transformations do not preserve a volume form!. Berezinian (superdeterminant) of an odd canonical transformation in general is not equal to unity

$$K = \begin{pmatrix} A & \mathscr{B} \\ \mathscr{C} & D \end{pmatrix}, \ \mathrm{Ber}\, K = \frac{\det\left(A - \mathscr{B} D^{-1} \mathscr{C}\right)}{\det D} \neq 1$$

Example

$$\mathrm{Ber} \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} = \frac{\det A}{\det D} = \frac{\det A}{\det (A^{+})^{-1}} = \det A^2, \ \text{since } A^{+} D = 1 \,.$$

Fact from linear algebra

Theorem

Let $K = \begin{pmatrix} A & \mathscr{B} \\ \mathscr{C} & D \end{pmatrix}$, be a matrix of a linear odd canonical transformation. Then

$$\mathrm{Ber}\, K = (\det A)^2, \sqrt{\mathrm{Ber}\, A} = \det A$$

Polynomial $\det A$ is a square root of Berezinian of odd canonical transformation K ("pfaffian of K").

$$K = K_1 K_2 = \begin{pmatrix} A_1 & \mathscr{B}_1 \\ \mathscr{C}_1 & D_1 \end{pmatrix} \begin{pmatrix} A_2 & \mathscr{B}_2 \\ \mathscr{C}_2 & D_2 \end{pmatrix} = \begin{pmatrix} A_1 A_2 + \mathscr{B}_1 \mathscr{C}_2 & \dots \\ \dots & \dots \end{pmatrix}$$

$$\mathrm{Ber}\, K = \mathrm{Ber}\, K_1 \mathrm{Ber}\, K_2$$

$$\det(A_1 A_2 + \mathscr{B}_1 \mathscr{C}_2) = \det A_1 \det A_2$$

Proof

$$K = \begin{pmatrix} A & \mathscr{B} \\ \mathscr{C} & D \end{pmatrix}: \qquad \begin{cases} A^{+}\mathscr{C} + \mathscr{C}^{+}A = 0 \\ D^{+}\mathscr{B} = \mathscr{B}^{+}D \\ A^{+}D + \mathscr{C}^{+}\mathscr{B} = 1 \end{cases}$$

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$$\begin{pmatrix} 1 + \mathscr{B}\mathscr{C} & \mathscr{B} \\ \mathscr{C} & 1 \end{pmatrix} \qquad \mathscr{B}^{+} = \mathscr{B}, \mathscr{C}^{+} = -\mathscr{C} .$$

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One can show that $\det(1 + \mathscr{B}\mathscr{C}) = 1$ since $\mathrm{Tr}^k(\mathscr{B}\mathscr{C}) = 0$

$$\mathrm{Ber}\, K = \mathrm{Ber} \begin{pmatrix} A & 0 \\ 0 & (A^+)^{-1} \end{pmatrix} \mathrm{Ber} \begin{pmatrix} 1 + \mathscr{B}\mathscr{C} & \mathscr{B} \\ \mathscr{C} & 1 \end{pmatrix}$$

$$= \frac{\det A}{\det (A^+)^{-1}} \frac{\det(1 + \mathscr{B}\mathscr{C} - \mathscr{B}\mathscr{C})}{\det 1} = \det A^2 .$$

The fact stated above underlines the deep geometrical properties of the odd Laplacian operator in Batalin Vilkovisky formalism.

Batalin-Vilkovisky Δ-operator

In 1981 I. Batalin and G. Vilkovisky considered the following second-order operator acting on functions on an odd symplectic superspace:

$$\Delta_0 F(x,\theta) = \frac{\partial^2 F(x,\theta)}{\partial x^a \partial \theta_a},$$

where (x^a, θ_a) are arbitrary Darboux coordinates on the odd symplectic superspace. This second order operator is invariant under arbitrary canonical transformations which preserve volume form $dx^1 \dots dx^n d\theta_1 \dots d\theta_n$

$$\underbrace{\{x^1,\dots,x^n;\theta_1,\dots,\theta_n\}}_{\text{Darboux coordinates}} \to \underbrace{\{\tilde{x}^1,\dots,\tilde{x}^n;\theta_1,\dots,\theta_n\}}_{\text{Darboux coordinates}} \text{ such that}$$

$$\mathrm{Ber}\,\frac{\partial(x',\theta')}{\partial(x,\theta)} = 1\,.$$

Batalin-Vilkovisky identity

For an arbitrary odd canonical transformation

$$\mathrm{Ber}\,\frac{\partial(x',\theta')}{\partial(x,\theta)} \neq 1\,.$$

This difference with an even canonical transformation is a reason why second order Laplacian arises.

On the other hand the following identity is obeyed:

$$\Delta_0 \sqrt{\left(\mathrm{Ber}\,\frac{\partial(x',\theta')}{\partial(x,\theta)}\right)} = 0\,.$$

This highly non-trivial identity obtained by Batalin and Vilkovisky is a core part of Δ-operators properties.

Invariant construction for BV Δ-operator

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$\rho = \rho(x,\theta) dx^1 \ldots dx^n d\theta_1 \ldots d\theta_n$—volume form

$D_F = \{f, x^a\}\frac{\partial}{\partial x^a} + \{f, \theta_a\}\frac{\partial}{\partial \theta_a}$—Hamiltonian vector field

$$\Delta_\rho = \Delta_0, \text{ if } \rho = 1\,.$$

(Kh. 1989)

Properties of $\Delta - operator$. BV master-equation

Let $\rho = \rho(x,\theta)dx^1 \dots dx^n d\theta_1 \dots d\theta_p$ be a volume form in odd symplectic superspace, ((x^i, θ_j) Darboux coordinates)

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Batalin-Vilkovisky master-equation for the master action $S=\log\sqrt{\rho}$.

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$$\Delta_\rho^2=0\,.$$

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Canonical odd laplacian on semidensities. Construction

Let M be an odd symplectic (super)manifold, i.e. $n|n$-dimensional (super)manifold endowed with an odd closed non-degenerate 2-form. The action of canonical odd Laplacian on an arbitrary semidensity $\mathbf{s} = s(x,\theta)\sqrt{dx^1 \dots dx^n d\theta_1 \dots d\theta_n}$ is defined by the formula

$$\Delta^\# \mathbf{s} = \frac{\partial^2 s(x,\theta)}{\partial x^a \partial \theta_a} \sqrt{dx^1 \dots dx^n d\theta_1 \dots d\theta_n}$$

where $\{x^1, \dots, x^n; \theta_1, \dots, \theta_n\}$ are an arbitrary Darboux coordinates on M.

Contrary to the Δ_ρ-operator on functions, the operator $\Delta^\#$ does not depend on volume form.
(Kh., 1999)

Spaces ΠT^*M and ΠTM

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Canonical (even) symplectic structure on T^*M:

$$\{x^i, p_j\} = \delta^i_j, \{x^i, x^j\} = 0, \{p_i, p_j\} = 0$$

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Change parity of fibres

$TM \to \Pi TM$ with coordinates (x^i, ξ^j),

$T^*M \to \Pi T^*M$ with coordinates (x^i, θ_j)
ΠT^*M is an odd symplectic supermanifold endowed with canonical odd symplectic structure:

$$\{x^i, \theta_j\} = \delta^i_j, \{x^i, x^j\} = 0, \{\theta_i, \theta_j\} = 0$$

$$\underbrace{F(x,\theta)}_{\text{function on } \Pi T^*M} = \underbrace{F(x) + F^i(x)\theta_i + F^{ij}\theta_i\theta_j + \cdots + F^{1\ldots n}\theta_1\ldots\theta_n}_{\text{mulitvector field on } M}$$

$$\underbrace{\omega(x,\xi)}_{\text{function on } \Pi TM} = \underbrace{\omega(x) + \omega_i(x)\xi^i + \omega_{ij}\xi^i\xi^j + \cdots + \omega_{1\ldots n}\xi^1\ldots\xi^n}_{\text{differential form on } M}$$

Differential form $\leftrightarrow$ Function on ΠT^*M?

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Differential form $\leftrightarrow$ Function on ΠT^*M? NO!

Differential form $\leftrightarrow$ semidensity on ΠT^*M

$$\omega(x,\xi)\xrightarrow{\tau}\mathbf{s}=\left(\int\omega(x,\xi)e^{\xi^i\theta_i}d\xi^1\dots d\xi^n\right)\sqrt{dx^1dx^2\dots dx^nd\theta_1\dots d\theta_n}$$

Example

Let $\omega=adx^1+bdx^2$ on M^2. Then

$$\mathbf{s}=\tau(\omega)=$$

$$\left(\int(a\xi^1+b\xi^2)e^{\xi^1\theta_1+\xi^2\theta_2}d\xi^1d\xi^2\right)\sqrt{dx^1dx^2d\theta_1d\theta_2}=$$

$$(a\theta_2-b\theta_1)\sqrt{dx^1dx^2d\theta_1d\theta_2}$$

semidensity on ΠT^*M^2.

Geometrical meaning of $\Delta^{\#}$

$$\begin{array}{ccc} \text{differential form on } M & \leftrightarrow & \text{semidensities on } \Pi T^* M \\ \downarrow & & \downarrow \\ \text{differential form on } M & \leftrightarrow & \text{semidensities on } \Pi T^* M \end{array}$$

$$\Delta^{\#}(\tau(\omega)) = \tau(d(\omega)),$$

$$d = \xi^i \frac{\partial}{\partial x^i}, \text{ exterior differential}$$

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Diffeomorphims of $M \subset$ canonical transformations of $\Pi T^* M$

Diffeomorphism of M defines canonical transformation of $\Pi T^* M$:

$$\tilde{x}^i = \tilde{x}^i(x^1, \dots, x^n) \to (\tilde{x}^i, \tilde{\theta}_j)\colon \begin{cases} \tilde{x}^i = \tilde{x}^i(x^1, \dots, x^n) \\ \tilde{\theta}_j = \frac{\partial x^m}{\partial x^j} \theta_m \end{cases} \qquad (*)$$

On the other hand a canonical transformation can be considered as a composition of transformation (*) and a special canonical transformation:

$$\begin{cases} \tilde{x}^i = x^i + f^i(x, \theta) \\ \tilde{\theta}_j = \theta_j + g(x, \theta) \end{cases} \quad \text{where } f^i(x, \theta)|_{\theta=0} = g^i(x, \theta)|_{\theta=0} = 0,$$

$$\sqrt{\mathrm{Ber}\, \frac{\partial(\tilde{x}, \tilde{\theta})}{\partial(x, \theta)}} = \det \frac{\partial \tilde{x}^i}{\partial x^j}$$

Compare this with decomposition for linear canonical transformation I

$$K = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} A' & \mathscr{B} \\ \mathscr{C} & 1 \end{pmatrix}$$

$$K = \begin{pmatrix} A & 0 \\ 0 & (A^{+})^{-1} \end{pmatrix} \begin{pmatrix} 1 + \mathscr{B}\mathscr{C} & \mathscr{B} \\ \mathscr{C} & 1 \end{pmatrix}$$

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One can show that $\det(1+\mathscr{B}\mathscr{C})=1$ since $\mathrm{Tr}^k(\mathscr{B}\mathscr{C})=0$

$$\mathrm{Ber}\,K=\mathrm{Ber}\begin{pmatrix} A & 0 \\ 0 & (A^+)^{-1} \end{pmatrix}\mathrm{Ber}\begin{pmatrix} 1+\mathscr{B}\mathscr{C} & \mathscr{B} \\ \mathscr{C} & 1 \end{pmatrix}$$

$$=\frac{\det A}{\det(A^+)^{-1}}\,\frac{\det(1+\mathscr{B}\mathscr{C}-\mathscr{B}\mathscr{C})}{\det 1}=\det A^2\,.$$

Question: How to describe canonical $\Delta^{\#}$ operator in invariant way?

(The original formula $\Delta^{\#}\mathbf{s} = \frac{\partial^2 s(x,\theta)}{\partial x^a \partial \theta_a} \sqrt{dx^1 \dots dx^p d\theta_1 \dots d\theta_p}$ is written in Darboux coordinates).

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In 2006 K. Bering wrote the explicit expression for $\Delta^{\#}$ operator in an arbitrary coordinates in terms of components of 2-form defining symplectic structure. He proved by straightforward calculations that this expression defines invariant operator which coincides with $\Delta^{\#}$-operator.

(See K. Bering "A Note on Semidensites in Antisymplectic Geometry".hep-th/0604)

Severa's spectral sequence

In 2005 P.Severa constructed the remarkable spectral sequence which contains as ingridients semidensites and $\Delta^{\#}$-operator. Thus he finds a natural definition of this 'somewhat miracolous operator'. (See P. Severa "On the origin of the BV operator..." (math/050633))

Let M be $n|n$-dimensional manifold with symplectic structure defined by odd non-degenerate closed two form ω.

Let $\Omega(M)$ be a space of all (pseudo)differential forms on M, i.e. functions on ΠTM.

Consider differential $Q = d + \omega$. For any F-function on ΠTM (differential form on E) $QF = dF + \omega F$.
One can see that

$$Q^2 = d^2 = \omega^2 = 0, d\omega + \omega d = 0$$

Spectral sequence $\{E_r, d_r\}$

$$E_{r+1} = H(E_r, d_r)$$

with $E_0 = \Omega(M)$, $d_0 = \omega$.

Theorem

The space $E_1 = H(\Omega(M), \omega)$ can be naturally identified with the space of semidensities on M.

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Elements of cohomology space $E_1 = H(\Omega(M), \omega)$ are represented in Darboux coordinates as classes $s(x, \theta)[dx^1 \dots dx^n]$. Under a change of Darboux coordinates $(x, \theta) \to (\tilde{x}, \tilde{\theta})$

$$[dx^1 \dots dx^n] \to \underbrace{\det\left(\frac{\partial x}{\partial \tilde{x}}\right)}$$

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with $E_0 = \Omega(M)$, $d_0 = \omega$.

Theorem

The space $E_1 = H(\Omega(M), \omega)$ can be naturally identified with the space of semidensities on M.

Elements of cohomology space $E_1 = H(\Omega(M), \omega)$ are represented in Darboux coordinates as classes $s(x,\theta)[dx^1 \dots dx^n]$. Under a change of Darboux coordinates $(x,\theta) \to (\tilde{x}, \tilde{\theta})$

$$[dx^1 \dots dx^n] \to \underbrace{\det\left(\frac{\partial x}{\partial \tilde{x}}\right)}_{\sqrt{\mathrm{Ber}\,\frac{\partial(\tilde{x},\tilde{\theta})}{\partial(x,\theta)}}} [d\tilde{x}^1 \dots d\tilde{x}^n]$$

Theorem

With identification of E_1 with semidensities the differential d_2 of the Severa's spectral sequence vanishes and differential d_3 coincides with the canonical operator $\Delta^{\#}$.
The spectral sequence degenerates at the term E_3.

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Remark Odd symplectic manifold is symplectomorphic to $\Pi T^* N$, where N is $(n, 0)$-dimensional Lagrangian surface in M. $Q = d + \omega$ is twisted differential:

$$QF = e^{-\Theta} d e^{\Theta} F,$$

where $d\Theta = \omega$, $(\Theta = \theta_a dx^a)$, Hence

$$H(Q, \Omega(M)) = H(d, M) = H_{\mathrm{de\,Rham}}(N)$$

A.Schwarz, I.Shapiro Twisted de Rham cohomology, homological definition of integral and ”Physics over ring” arXiv;0809.0086 [math.AG]

Thank you